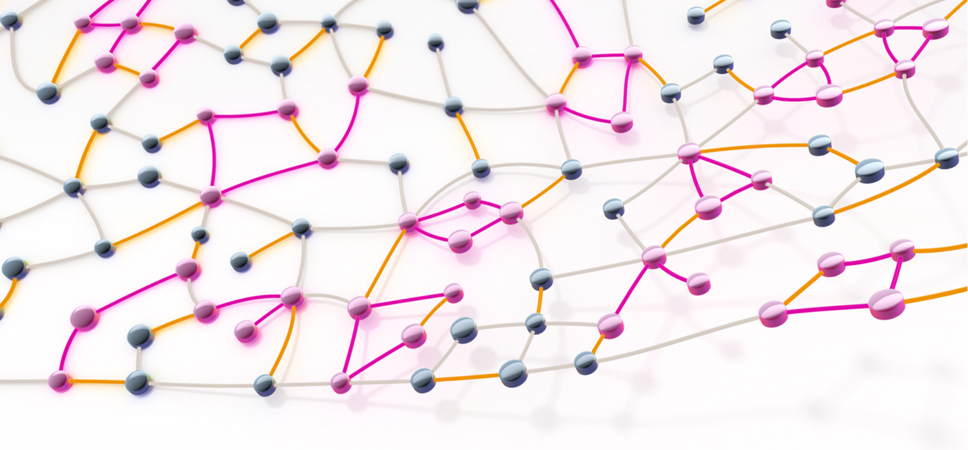
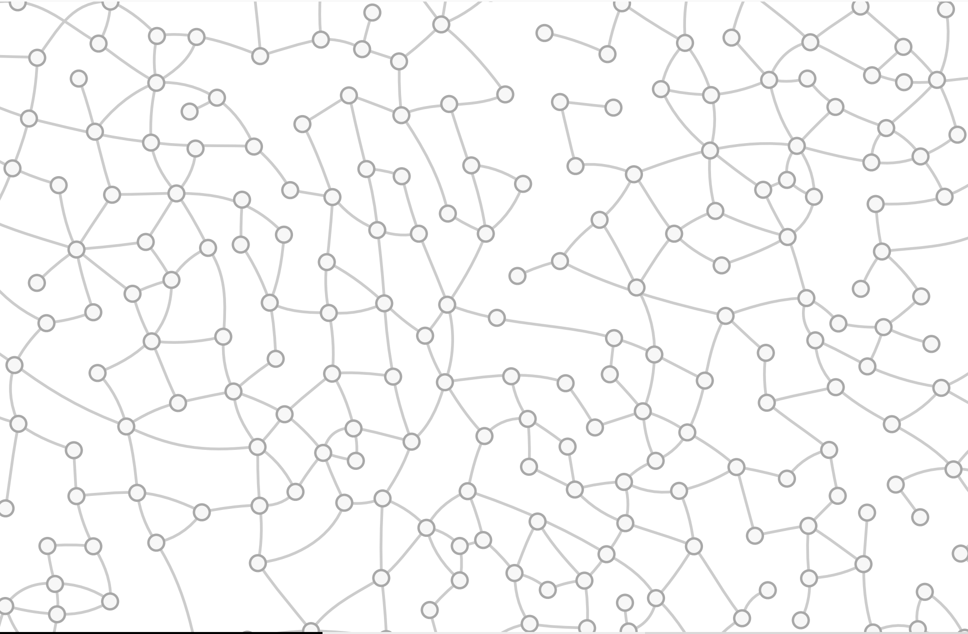


Bipartite Vertex Cover

Mika Göös — *University of Toronto & HIIT*
Jukka Suomela — *University of Helsinki & HIIT*

$\mathcal{LOCAL}$ model



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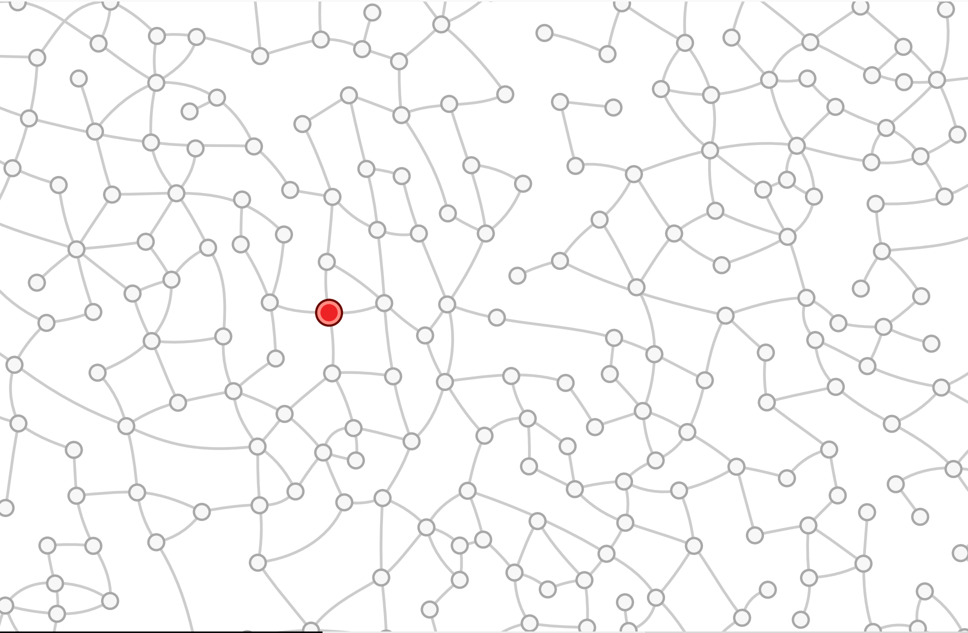

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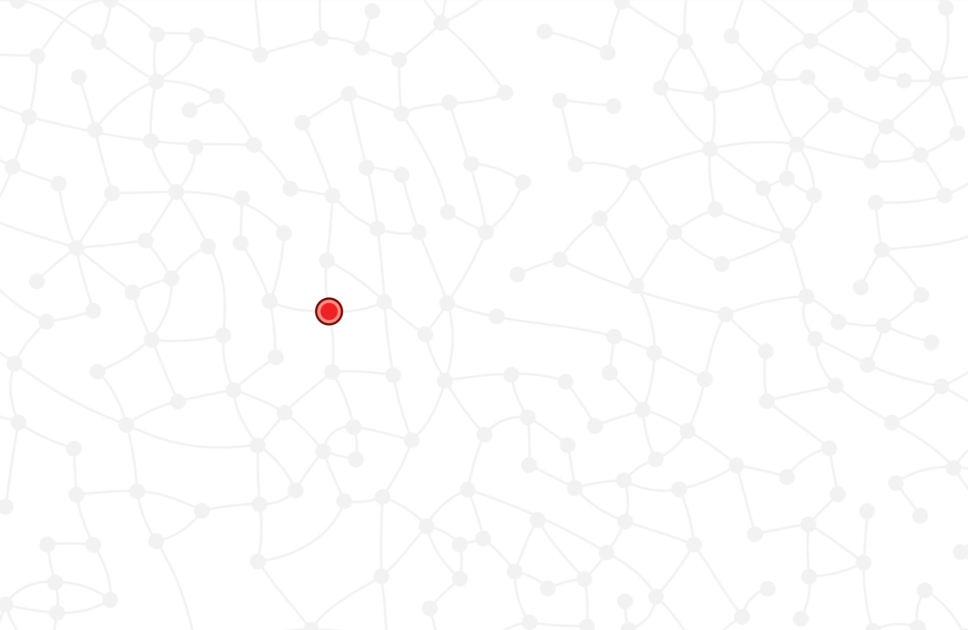

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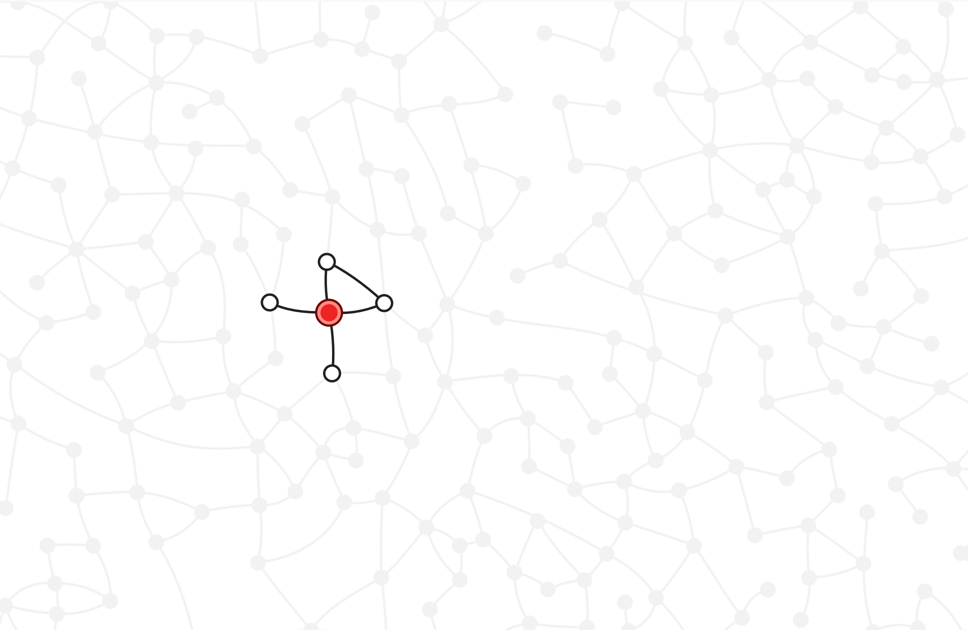

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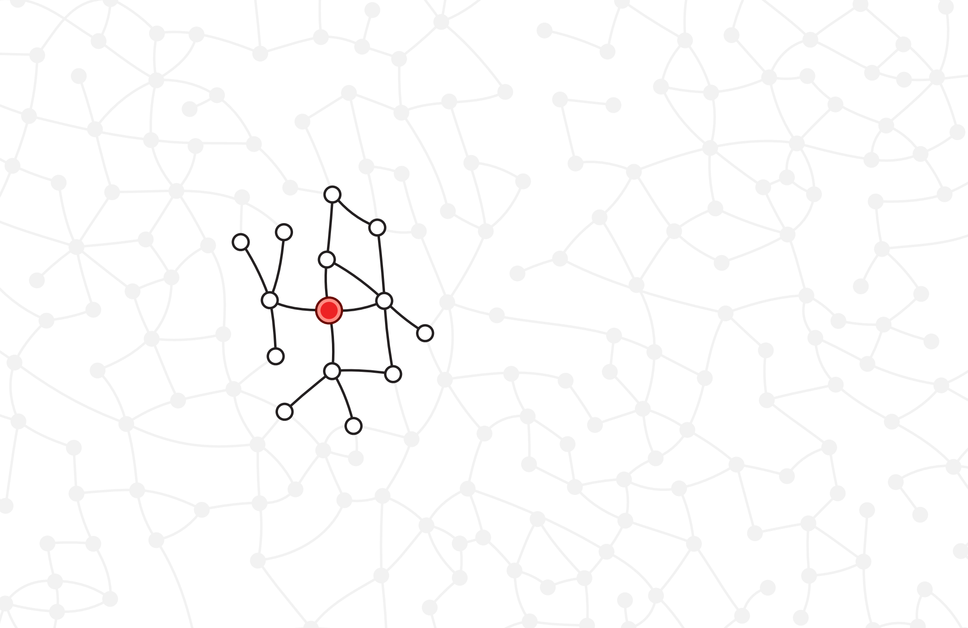

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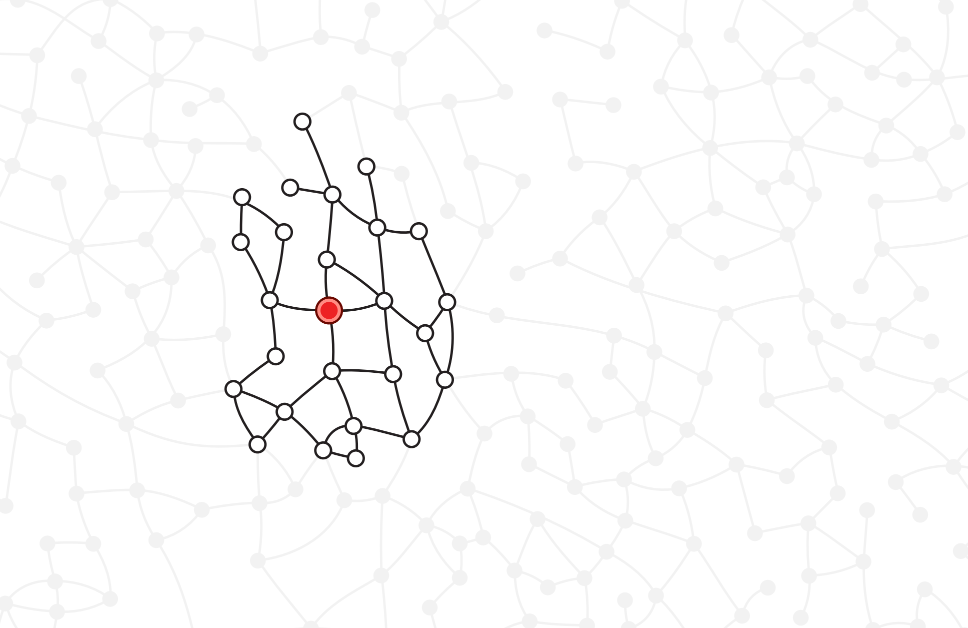

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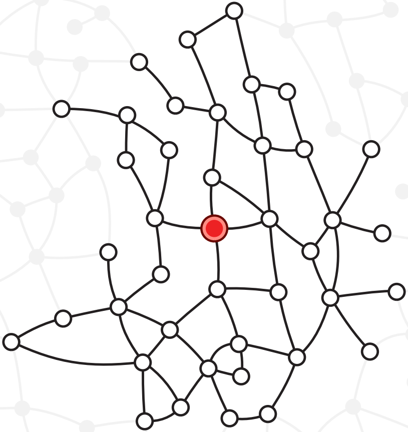

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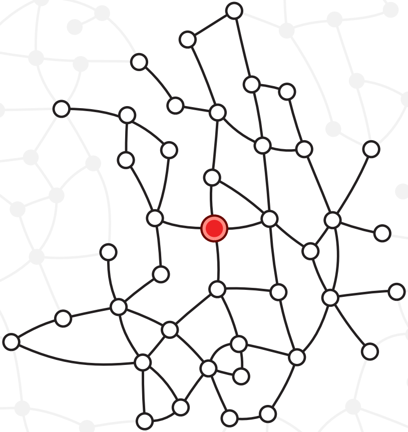
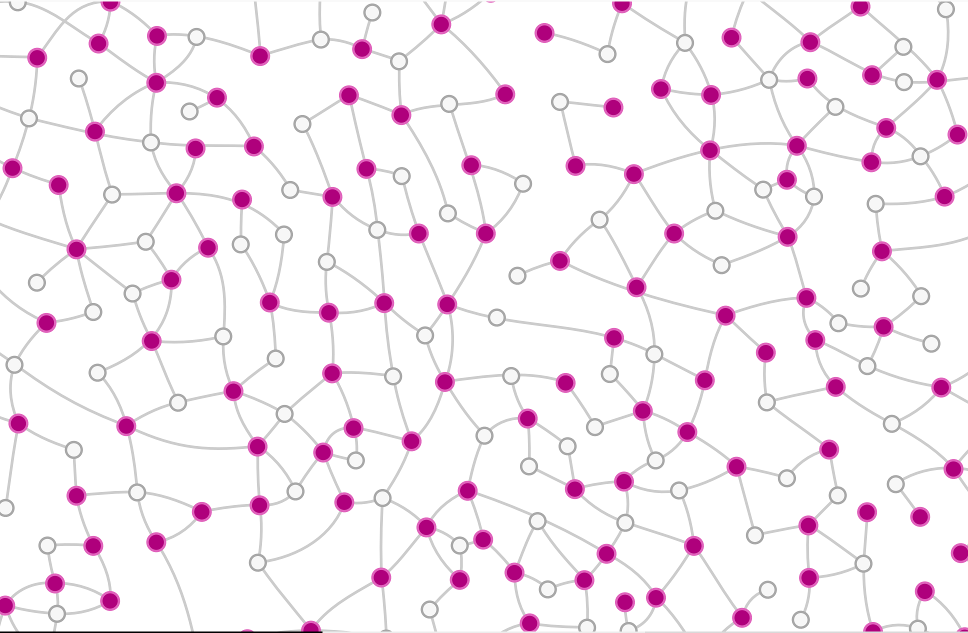


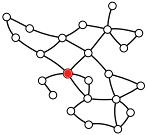
$$\mapsto \{0,1\}$$

$\mathcal{LOCAL}$ model



$\mathcal{LOCAL}$ model

Definition:

$\mathbf{A} : \{$  $\} \to \{0, 1\}$

Run-time R
= radius-R neighbourhood:

1. Nodes have **unique IDs**
2. Nodes get **random strings** as input

Prior work on **Min Vertex Cover** (MIN-VC)

	Apx ratio	**Run-time**	
General graphs	$O(1)$	$\Omega(\sqrt{\log n})$	[KMW **PODC'04**]

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Note: MIN-VC is solvable on **bipartite** graphs using sequential polynomial-time algorithms!

The **bipartite** case

Question: Can we approximate MIN-VC fast on **bipartite** graphs?

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$(1+\epsilon)$-approximation scheme?

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Setting:
- Bipartite **2-coloured** graph
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Integer	**Min** VC	**Max** Matching
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= LP duality

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Integer	**Min** VC	=	**Max** Matching
	$\Vert$		$\Vert$
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= LP duality
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The **bipartite** case

Setting:
- Bipartite **2-coloured** graph
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	Covering	Packing
Integer	**Min** VC	**Max** Matching
LP	$O_\epsilon(1)$	$O_\epsilon(1)$

[KMW **SODA'06**]

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Setting:
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- Compute $(1+\epsilon)$-approximation

	Covering	Packing
Integer	**Min** VC	$O_\epsilon(1)$
LP	$O_\epsilon(1)$	$O_\epsilon(1)$

[KMW **SODA'06**]
[NO **FOCS'08**], [ÅPRSU **'10**]

The **bipartite** case

	Covering	Packing
Integer	???	$O_\epsilon(1)$
LP	$O_\epsilon(1)$	$O_\epsilon(1)$

The **bipartite** case

	Covering	**Packing**
Integer	$\Omega(\log n)$	$O_\epsilon(1)$
LP	$O_\epsilon(1)$	$O_\epsilon(1)$

The **bipartite** case

Surprise: **No Sublogarithmic-Time Approximation Scheme for Bipartite Vertex Cover!**

	Covering	**Packing**
Integer	$\Omega(\log n)$	$O_\epsilon(1)$
LP	$O_\epsilon(1)$	$O_\epsilon(1)$

Our result

Main Theorem

$\exists\delta > 0$: No $o(\log n)$-time algorithm to $(1+\delta)$-approximate MIN-VC on 2-coloured graphs of max degree $\Delta = 3$

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Lower bound is tight

1. There is $O_\epsilon(\log n)$-time approx. scheme [LS '93]
2. If $\Delta = 2$ there is $O_\epsilon(1)$-time approx. scheme

Why is **MIN-VC** difficult for distributed graph algorithms?

Short answer: Solving MIN-VC requires solving a hard **cut minimisation** problem

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Short answer: Solving MIN-VC requires solving a hard **cut minimisation** problem

Strategy:
1. Reduce cut problem to MIN-VC
2. Prove that cut problem is hard

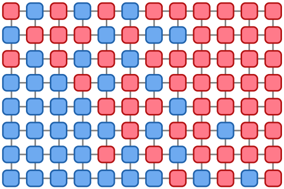
Reduction formalised

RECUT problem

Input: Labelled graph (G, ℓ_{in}) where $\ell_{\mathsf{in}}\colon V \to \{\mathsf{red}, \mathsf{blue}\}$
Output: Labelling $\ell_{\mathsf{out}}\colon V \to \{\mathsf{red}, \mathsf{blue}\}$ that minimises the size of the cut $|\ell_{\mathsf{out}}|$ subject to

- If ℓ_{in} is all-red then ℓ_{out} is all-red
- If ℓ_{in} is all-blue then ℓ_{out} is all-blue

ℓ_{in}



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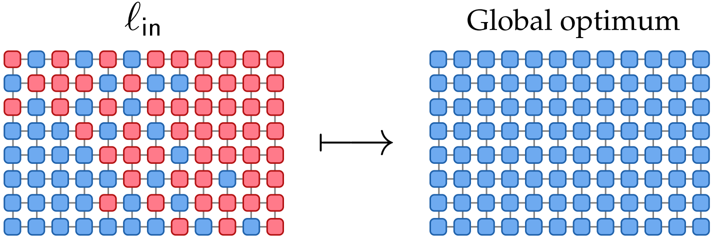

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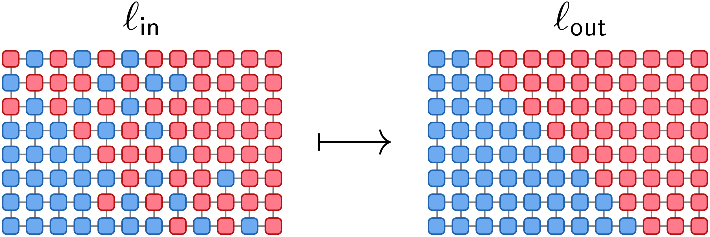

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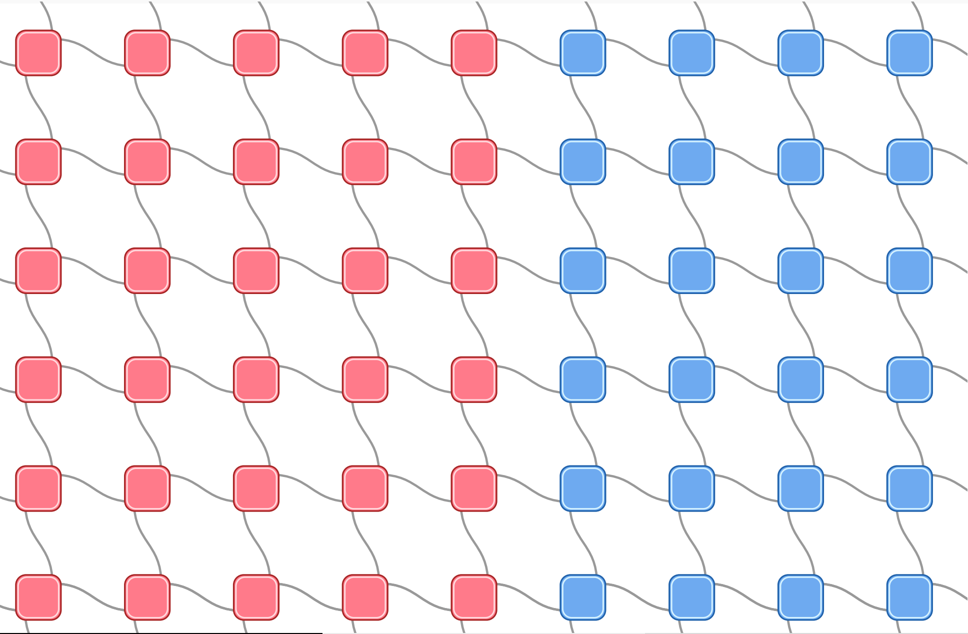
Recut $\leq$ Min-VC

If: Min-VC can be $(1+\epsilon)$-approximated in time R
Then: We can compute in time R a Recut of density

$$\frac{|\ell_{\mathsf{out}}|}{|E|} \leq \epsilon$$

Reduction in pictures



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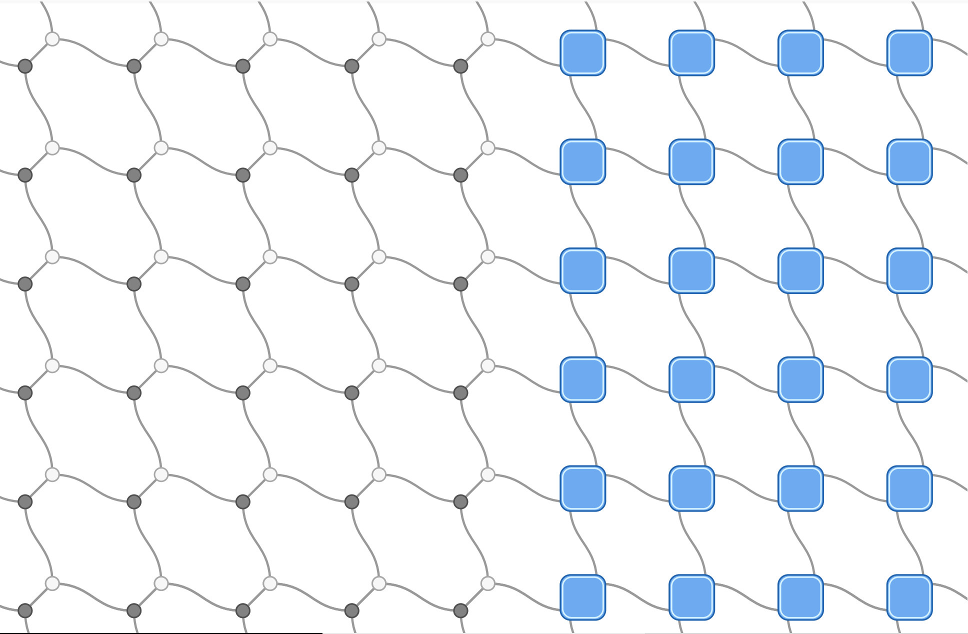

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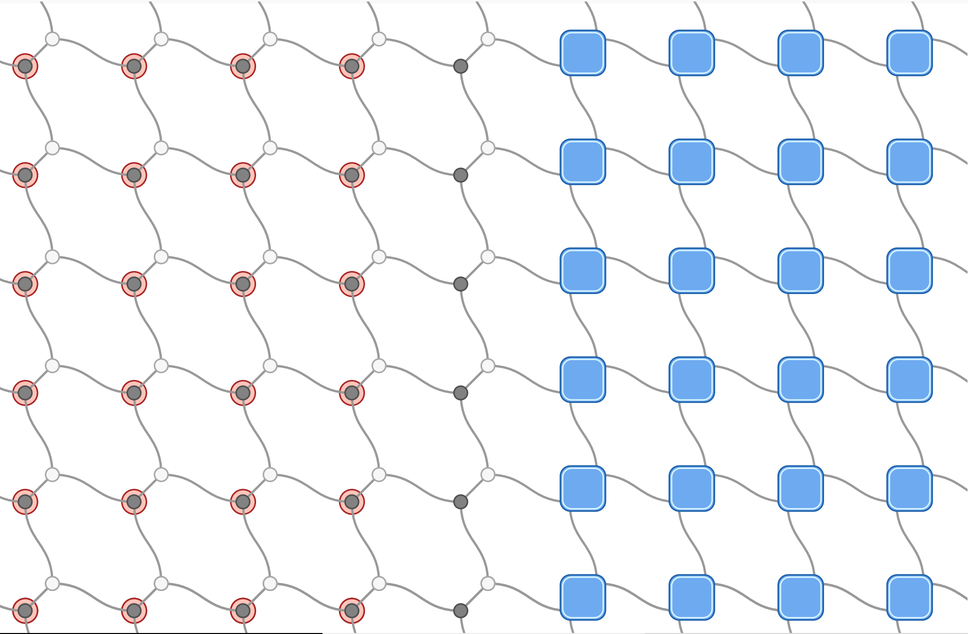

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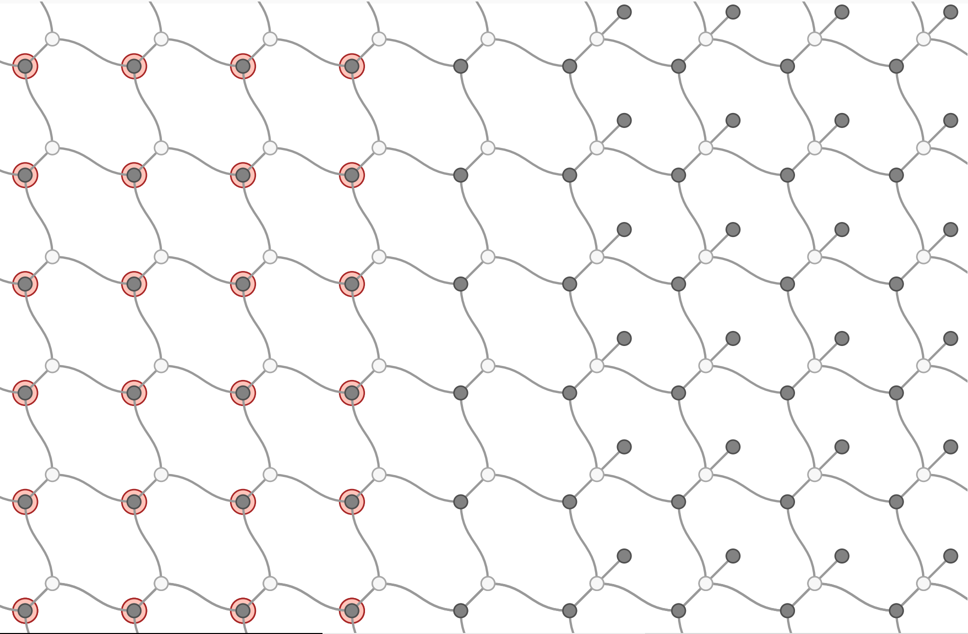

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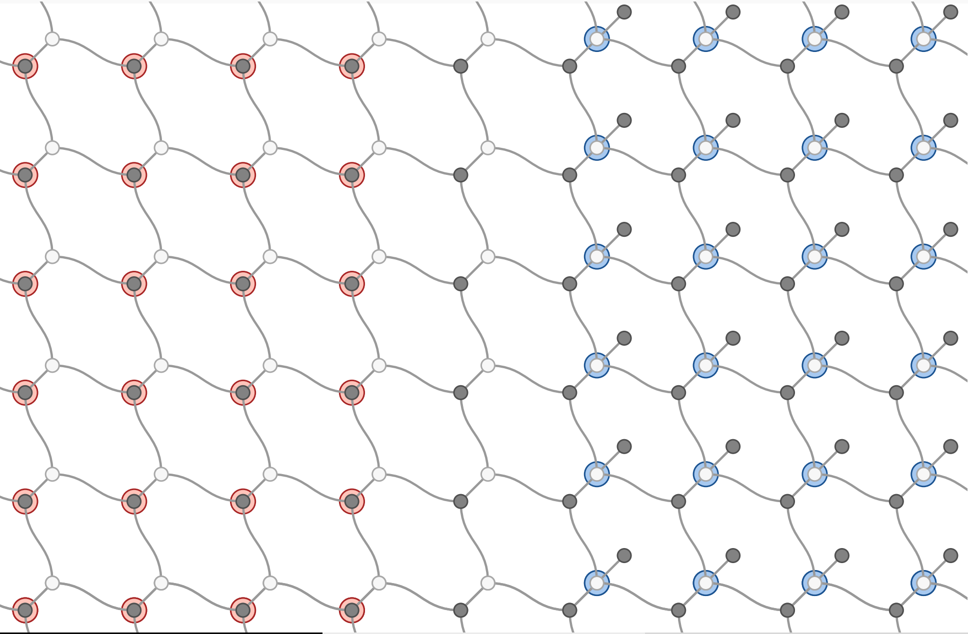

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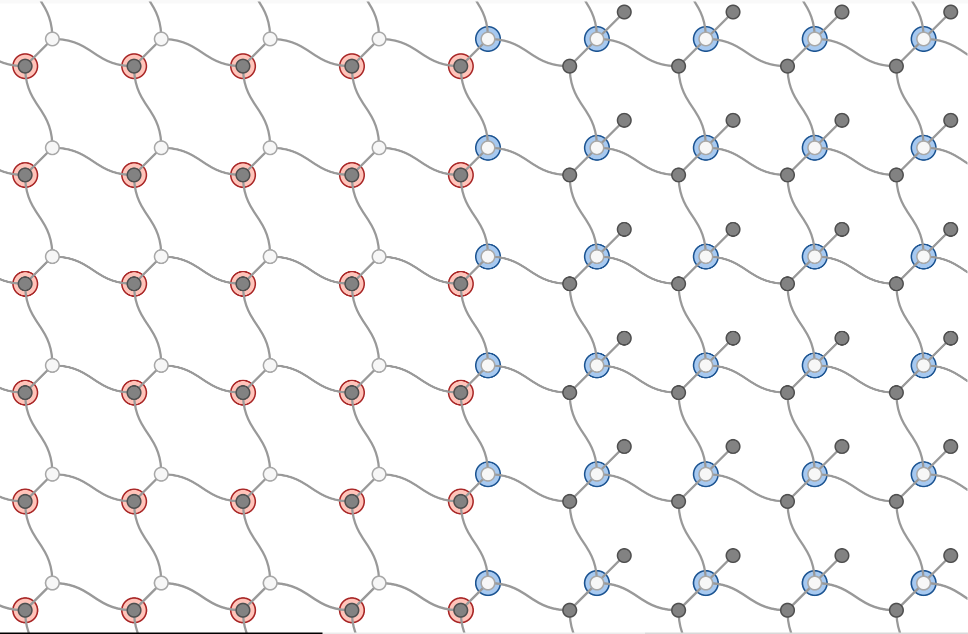

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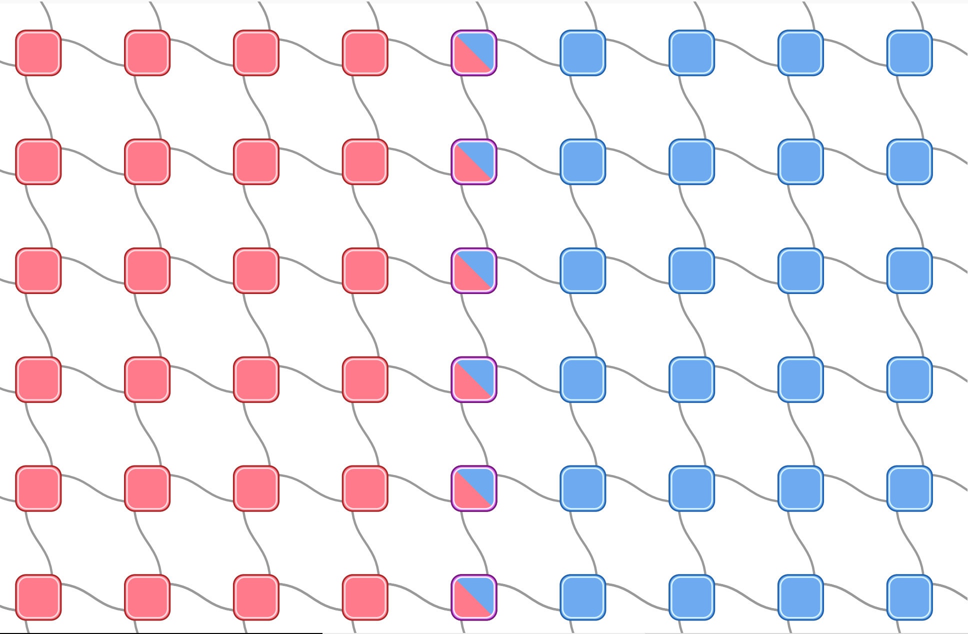

On RECUT

Theorem: RECUT $\leq$ MIN-VC

On Recut

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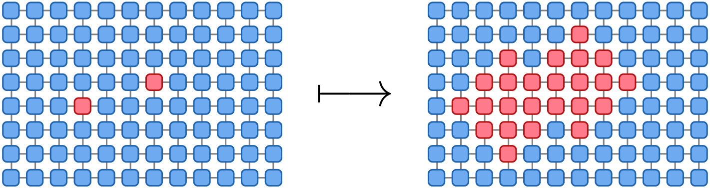
Sometimes Recut Is Easy:
The algorithm "Output red iff you see any red nodes" computes a small Recut on grid-like graphs

ℓ_{in} $\longmapsto$ ℓ_{out}



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Therefore: We consider **expander graphs** that satisfy

$$|\ell| \geq \delta \cdot \min(|\ell^{-1}(\mathsf{red})|, |\ell^{-1}(\mathsf{blue})|)$$

On ReCut

Theorem: ReCut $\leq$ Min-VC

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Main Technical Lemma: We fool a fast algorithm into producing a **balanced** ReCut

$$|\ell_{\mathsf{out}}^{-1}(\mathsf{red})| \approx |\ell_{\mathsf{out}}^{-1}(\mathsf{blue})| \approx n/2$$

Conclusions

Our result:

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Cheers!