Communication Complexity

vs.

Partition Numbers

(Based on 2 papers to appear at FOCS)

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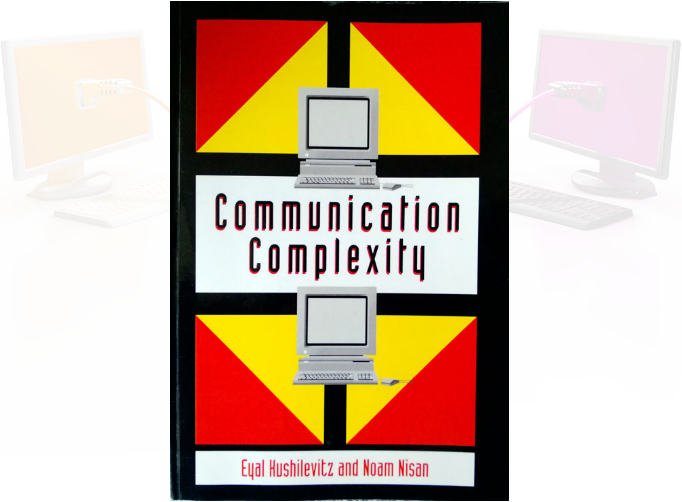
Communication
Complexity
Eyal Kushilevitz and Noam Nisan

Chapter 1 . . .

Communication complexity

[Yao, ’79]

Alice

$x \in \{0,1\}^n$

Bob

$y \in \{0,1\}^n$

Compute: $F(x,y) \in \{0,1\}$

Deterministic protocols

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1

$$F\colon \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

Deterministic protocols

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
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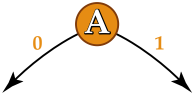
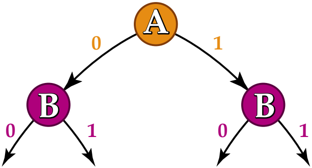
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1



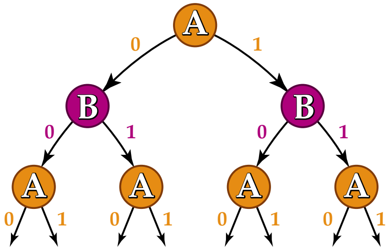
Deterministic protocols

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1



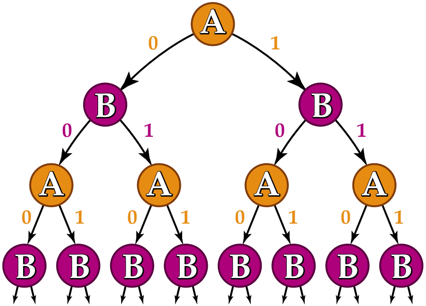
Deterministic protocols

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1



Deterministic protocols

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1



Deterministic protocols

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
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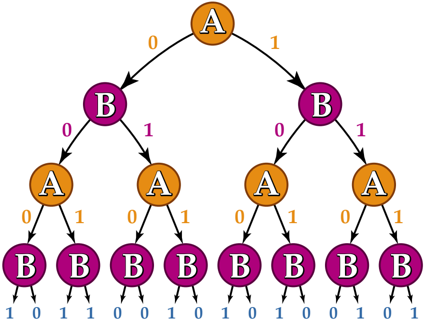
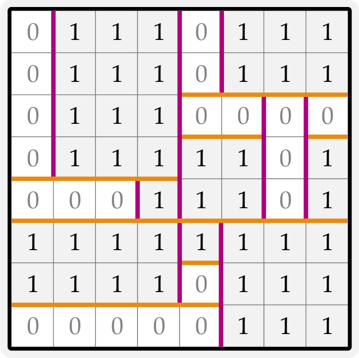
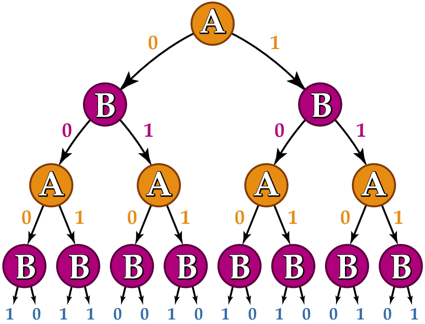
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1



Deterministic protocols





- **P**$^{\mathbf{cc}}(F)$:= Deterministic communication complexity of F
- **Partition number** $\chi(F)$:= Least number of monochromatic rectangles required to partition the communication matrix

Deterministic protocols

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
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0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1

Basic fact:

$$\mathbf{P}^{\mathbf{cc}}(F) \;\geq\; \log\chi(F)$$

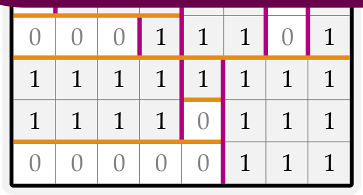
[Kushilevitz–Nisan]:

$$\mathbf{P}^{\mathbf{cc}}(F) \leq O(\log\chi(F))\ ?$$

- $\mathbf{P}^{\mathbf{cc}}(F)$:= Deterministic communication complexity of F
- **Partition number** $\chi(F)$:= Least number of monochromatic rectangles required to partition the communication matrix

Our results

- *Theorem 1:* $\exists F: \quad \mathbf{P}^{\mathrm{cc}}(F) \geq \tilde{\Omega}(\log^{1.5} \chi(F))$



[Kushilevitz–Nisan]:

$\mathbf{P}^{\mathrm{cc}}(F) \leq O(\log \chi(F))$?

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- *Previously:* [Aho–Ullman–Yannakakis, STOC'83]:
  $\forall F: \quad \mathbf{P}^{\mathbf{cc}}(F) \;\leq\; O(\log^2\chi(F))$

  [Kushilevitz–Linial–Ostrovsky, STOC'96]:
  $\exists F: \quad \mathbf{P}^{\mathbf{cc}}(F) \;\geq\; 2\cdot\log\chi(F)$

Our results

- *Theorem 1:* $\exists F : \quad \mathbf{P}^{\mathbf{cc}}(F) \geq \tilde{\Omega}(\log^{1.5} \chi(F))$

- *Theorem 2:* $\exists F : \quad \mathbf{P}^{\mathbf{cc}}(F) \geq \tilde{\Omega}(\log^{2} \chi_1(F))$

One-sided *partition numbers:*

$\chi(F) = \chi_1(F) + \chi_0(F)$, $\qquad \chi_i(F) :=$ least number of rectangles needed to partition $F^{-1}(i)$

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- *Theorem 1:* $\exists F: \quad \mathbf{P}^{\mathbf{cc}}(F) \ \geq \ \tilde{\Omega}(\log^{1.5}\chi(F))$

- *Theorem 2:* $\exists F: \quad \mathbf{P}^{\mathbf{cc}}(F) \ \geq \ \tilde{\Omega}(\log^{2}\chi_1(F))$
- *Previously:* *Clique vs. Independent Set* [Yannakakis, STOC'88]:
  $\forall F: \quad \mathbf{P}^{\mathbf{cc}}(F) \ \leq \ O(\log^{2}\chi_1(F))$

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- *Corollary:* $\exists F: \quad \mathbf{P}^{\mathrm{cc}}(F) \geq \tilde{\Omega}(\log^{2} \mathrm{rank}(F))$

Observation: $\chi_1(F) \geq \mathrm{rank}(F)$

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- *Corollary:* $\exists F: \quad \mathbf{P}^{\mathbf{cc}}(F) \ \geq\ \tilde{\Omega}(\log^{2}\mathrm{rank}(F))$

- *Previously:* *Log-rank conjecture* [Lovász–Saks, FOCS'88]:
  $\forall F: \quad \mathbf{P}^{\mathbf{cc}}(F) \ \leq\ \log^{O(1)}\mathrm{rank}(F)$

  [Kushilevitz–Nisan–Wigderson, FOCS'94]:
  $\exists F: \quad \mathbf{P}^{\mathbf{cc}}(F) \ \geq\ \Omega(\log^{1.63}\mathrm{rank}(F))$

Our results

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- *Corollary:* $\exists F: \quad \mathbf{P}^{\mathrm{cc}}(F) \geq \tilde{\Omega}(\log^{2} \mathrm{rank}(F))$

- *Theorem 3:* $\exists F: \quad \mathbf{coNP}^{\mathrm{cc}}(F) \geq \Omega(\log^{1.128} \chi_1(F))$

  $\|$

  Co-nondeterministic communication complexity

Nondeterministic protocols

Algorithmic definition:

1. Players **guess** a proof string $p \in \{0,1\}^C$
2. **Alice** accepts depending on (x, p)
3. **Bob** accepts depending on (y, p)
4. (x, y) is **accepted** iff both players accept

$\mathbf{NP^{cc}}(F)$:= Least C for which there is an above type protocol accepting $F^{-1}(1)$

Combinatorial definition:

$\mathbf{NP^{cc}}(F)$:= $\log \mathrm{Cov}_1(F)$

$\mathrm{Cov}_1(F)$:= Least number of monochromatic rectangles needed to cover $F^{-1}(1)$

Unambiguity: At most one accepting proof

Nondeterministic protocols

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1

$$\mathbf{NP^{cc}} = \log \mathrm{Cov}_1(F)$$

$$\mathbf{UP^{cc}} = \log \chi_1(F)$$

$$\mathbf{2UP^{cc}} = \log \chi(F)$$

Nondeterministic protocols

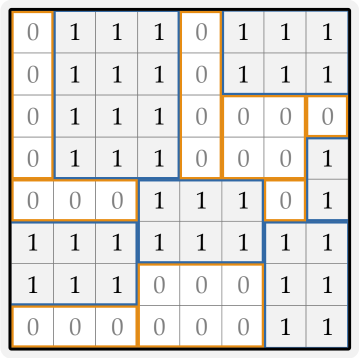
0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	0	0	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	0	0	0	1	1
0	0	0	0	0	0	1	1

$\mathbf{NP}^{\mathbf{cc}} = \log \mathrm{Cov}_1(F)$

$\mathbf{UP}^{\mathbf{cc}} = \log \chi_1(F)$

$\mathbf{2UP}^{\mathbf{cc}} = \log \chi(F)$

UP = Unambiguous **NP**



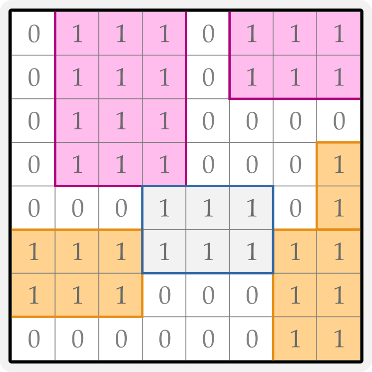
$\mathbf{NP}^{\mathbf{cc}} = \log \mathrm{Cov}_1(F)$

$\mathbf{UP}^{\mathbf{cc}} = \log \chi_1(F)$

$\mathbf{2UP}^{\mathbf{cc}} = \log \chi(F)$

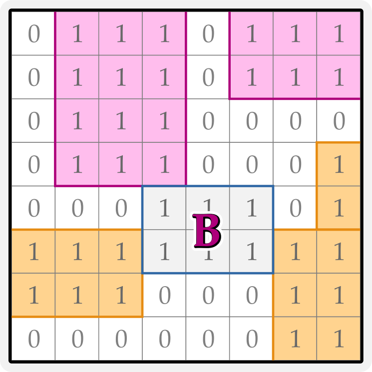
Shorthand: $\mathbf{2UP} = \mathbf{UP} \cap \mathbf{coUP}$

[Yannakakis, STOC'88]: $\mathbf{P}^{\mathbf{cc}} \le (\mathbf{UP}^{\mathbf{cc}})^2$



A rectangle is **good** for **Alice** (**Bob**) if at most half of the other rectangles intersect it in **rows** (**columns**)

[Yannakakis, STOC'88]: $\mathbf{P}^{\mathbf{cc}} \leq (\mathbf{UP}^{\mathbf{cc}})^2$



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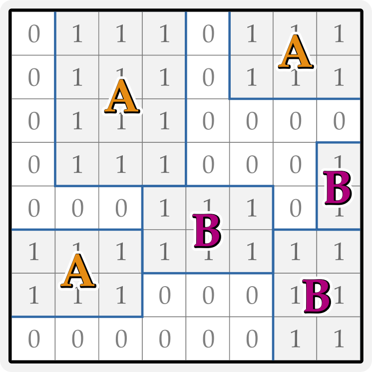

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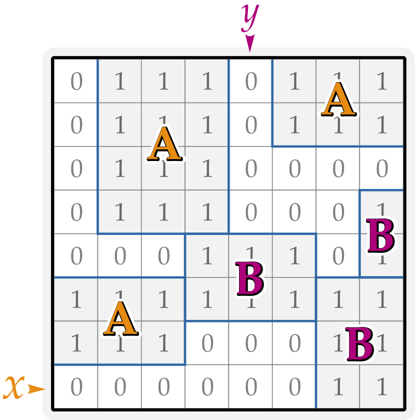
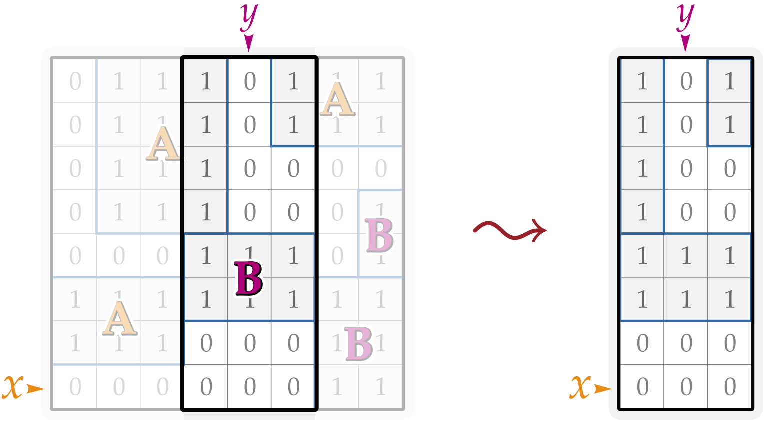


Players announce a name of a **good-for-them** rectangle that intersects their row/column

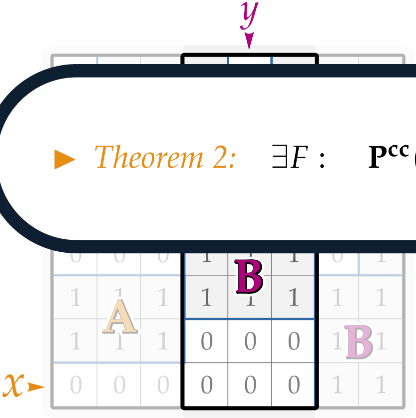
[Yannakakis, STOC'88]: $\mathbf{P}^{\mathbf{cc}} \leq (\mathbf{UP}^{\mathbf{cc}})^2$



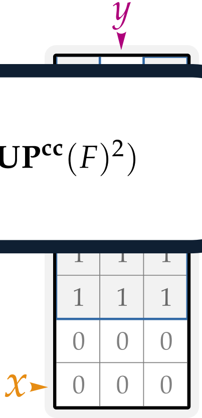
$\mathbf{UP}^{\mathbf{cc}} = k$ $\qquad$ $\mathbf{UP}^{\mathbf{cc}} = k - 1$

[Yannakakis, STOC'88]: $\mathbf{P}^{\mathbf{cc}} \leq (\mathbf{UP}^{\mathbf{cc}})^2$

- *Theorem 2:* $\exists F: \quad \mathbf{P}^{\mathbf{cc}}(F) \geq \tilde{\Omega}(\mathbf{UP}^{\mathbf{cc}}(F)^2)$



$\mathbf{UP}^{\mathbf{cc}} = k$



$\mathbf{UP}^{\mathbf{cc}} = k - 1$

Our results

- *Theorem 1:* $\exists F: \quad \mathbf{P}^{\mathbf{cc}}(F) \geq \tilde{\Omega}(\mathbf{2UP}^{\mathbf{cc}}(F)^{1.5})$

- *Theorem 2:* $\exists F: \quad \mathbf{P}^{\mathbf{cc}}(F) \geq \tilde{\Omega}(\mathbf{UP}^{\mathbf{cc}}(F)^{2})$

- *Theorem 3:* $\exists F: \quad \mathbf{coNP}^{\mathbf{cc}}(F) \geq \Omega(\mathbf{UP}^{\mathbf{cc}}(F)^{1.128})$

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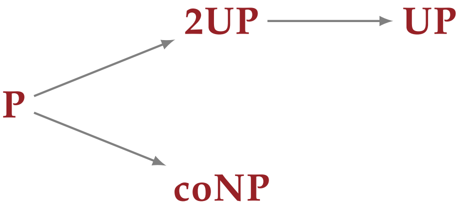

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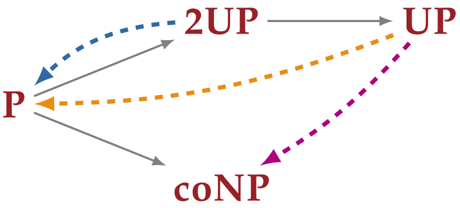


Clique vs. Independent Set

[Yannakakis, STOC'88]:

Clique vs. Independent Set is **complete** for $\mathbf{UP}^{\mathbf{cc}}$

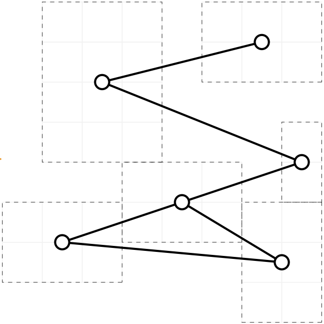
CIS_G on a graph $G = (V, E)$:

- **Alice** holds a clique $C \subseteq V$
- **Bob** holds an independent set $I \subseteq V$
- Output $|C \cap I| \in \{0, 1\}$

Note: $\mathbf{UP}^{\mathbf{cc}}(\text{CIS}_G) \leq \log |V|$

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	0	0	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	0	0	0	1	1
0	0	0	0	0	0	1	1

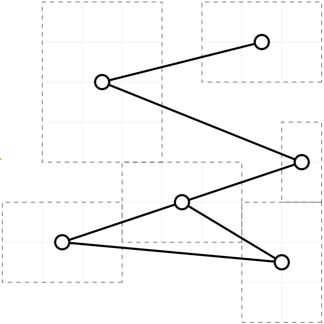
$\rightsquigarrow$



Fix a partition of $F^{-1}(1)$. Construct $G = (V, E)$ where $V = \{\text{rectangles}\}$ and $\{u, v\} \in E$ iff u and v share a row

***F* reduces to CIS$_G$:** **Alice** (**Bob**) maps her **row** (**column**) to the set of rectangles intersecting it

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	0	0	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	0	0	0	1	1
0	0	0	0	0	0	1	1



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Summary: $\mathbf{UP^{cc}}(F) = \mathbf{UP^{cc}}(\mathrm{CIS}_G) = \log |V|$

More on CIS

Yannakakis's motivation:

Size of LPs for the vertex packing polytope of G

Breakthrough: [Fiorini et al., STOC'12]

For an *n*-node graph:

$$\begin{aligned} \forall G: \quad & \mathbf{UP^{cc}}(\mathrm{CIS}_G) = \log n \\ \forall G: \quad & \mathbf{P^{cc}}(\mathrm{CIS}_G) \le O(\log^2 n) \\ \forall G: \quad & \mathbf{coNP^{cc}}(\mathrm{CIS}_G) \le O(\log^2 n) \end{aligned}$$

Yannakakis's question:

$$\forall G: \quad \mathbf{coNP^{cc}}(\mathrm{CIS}_G) \le O(\log n) \quad ?$$

More on CIS

Alon–Saks–Seymour conjecture:

$\forall G:$ $\mathrm{chr}(G) \leq \mathrm{bp}(G) + 1$?

$\mathrm{chr}(G)$:= Chromatic number of G
$\mathrm{bp}(G)$:= Least number of bicliques needed to partition $E(G)$

Yannakakis's question:

$\forall G:$ $\mathbf{coNP}^{\mathbf{cc}}(\mathrm{CIS}_G) \leq O(\log n)$?

More on CIS

Alon–Saks–Seymour conjecture:

$\forall G$:   ~~$\mathrm{chr}(G) \leq \mathrm{bp}(G) + 1$~~   ?

[Huang–Sudakov, 2010]: $\exists G$: $\mathrm{chr}(G) \geq \mathrm{bp}(G)^{6/5}$
[Shigeta–Amano, 2014]: $\exists G$: $\mathrm{chr}(G) \geq \mathrm{bp}(G)^{2}$

Yannakakis's question:

$\forall G$: $\mathbf{coNP}^{\mathbf{cc}}(\mathrm{CIS}_G) \leq O(\log n)$?

More on CIS

Polynomial Alon–Saks–Seymour conjecture:

$\forall G:$ $\qquad \mathrm{chr}(G) \le \mathbf{poly}(\mathrm{bp}(G))$?

Yannakakis's question:

$\forall G:$ $\qquad \mathbf{coNP^{cc}}(\mathrm{CIS}_G) \le O(\log n)$?

More on CIS

Polynomial Alon–Saks–Seymour conjecture:

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[Alon–Haviv] $\Downarrow$ $\Uparrow$ [Bousquet et al.]

Yannakakis's question:

$\forall G$: $\mathbf{coNP}^{\mathbf{cc}}(\mathrm{CIS}_G) \leq O(\log n)$?

More on CIS

Po[redacted]ynomial Alon–Saks–Seymour conj[redacted]re:

∀G : [redacted]$(G) \;\leq\;$ **poly**$($bp$(G$[redacted] ?

[Alon–Haviv] ⇓   ⇑ [Bousquet et al.]

Yannakak[redacted]question:

∀G[redacted] $\mathbf{coNP^{cc}}(\mathrm{CIS}_G) \;\leq\; O(\log n)$?

More on CIS

Polynomial Alon–Saks–Seymour conjecture:

$\forall G: \quad \chi(G) \le \mathbf{poly}(\mathrm{bp}(G))$?

- *Theorem 3:* $\exists F: \quad \mathbf{coNP^{cc}}(F) \ge \Omega(\mathbf{UP^{cc}}(F)^{1.128})$

Yannakakis' question:

$\forall G: \quad \mathbf{coNP^{cc}}(\mathrm{CIS}_G) \le O(\log n)$?

Communication:

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- *Theorem 3:* $\exists F : \quad \mathbf{coNP}^{\mathbf{cc}}(F) \ \geq\ \Omega(\mathbf{UP}^{\mathbf{cc}}(F)^{1.128})$

2-step strategy:

- Prove analogous **query** separations
- Apply a **communication↔query** simulation theorem

Decision tree:

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2-step strategy:

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Step 1: Query separations

Decision tree models

$$f\colon \{0,1\}^n \to \{0,1\}$$

$\mathbf{P}^{\mathbf{dt}}(f)$ = **Deterministic** query complexity

$\mathbf{NP}^{\mathbf{dt}}(f)$ = **Nondeterministic** query complexity
= 1-certificate complexity
= DNF width

$\mathbf{UP}^{\mathbf{dt}}(f)$ = **Unambiguous** query complexity
= Unambiguous DNF width

Quadratic **P**-vs-**UP** gap

Warm-up example f:

- M is $k \times k$ matrix with entries in $\{0,1\}$
- $f(M) = 1 \iff M$ contains a **unique all-1 column**

0	1	1	1	1	1
1	0	0	1	1	1
0	1	0	1	0	1
1	0	1	1	1	0
1	1	1	1	1	0
1	1	0	1	1	1

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			1		
			1		
0			1	0	
	0		1		
			1		0
		0	1		

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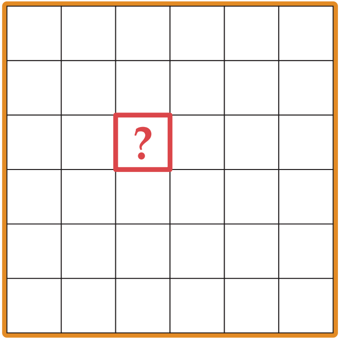

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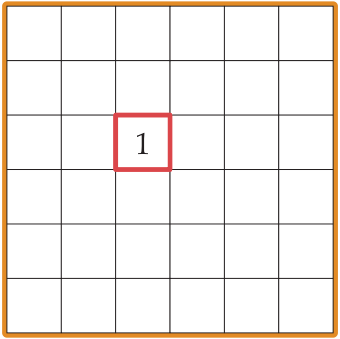

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1		1	1	1	1
		1			1
					?
	1				1
1	1		1	1	1

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					0
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1	1	1	1	1	1
0	1	1	1	0	1
1	0	1	1	1	0
1	1	1	?	1	1
1	1	0	1	1	1

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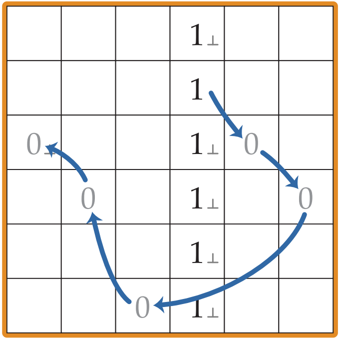
Actual gap example f:

- M is $k \times k$ matrix with entries in $\{0,1\} \times ([k] \times [k] \cup \{\bot\})$
- $f(M) = 1 \iff M$ contains a **unique all-1 column** that has a **linked list** through 0's in other columns



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$1_\perp$	$1_\perp$	$1_\perp$			$1_\perp$
$1_\perp$					$1_\perp$
?					
$1_\perp$					
$1_\perp$				$1_\perp$	$1_\perp$
$1_\perp$	$1_\perp$		$1_\perp$	$1_\perp$	$1_\perp$

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$1_\perp$	$1_\perp$	$1_\perp$			$1_\perp$
$1_\perp$					$1_\perp$
$0_\perp$					
$1_\perp$					
$1_\perp$				$1_\perp$	$1_\perp$
$1_\perp$	$1_\perp$		$1_\perp$	$1_\perp$	$1_\perp$

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$1_\perp$	$1_\perp$	$1_\perp$			$1_\perp$
$0_\perp$	$1_\perp$				
$1_\perp$	?			$1_\perp$	
$1_\perp$	$1_\perp$			$1_\perp$	$1_\perp$
$1_\perp$	$1_\perp$		$1_\perp$	$1_\perp$	$1_\perp$

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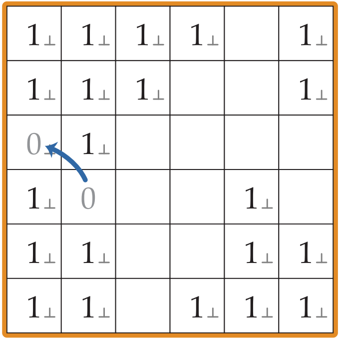

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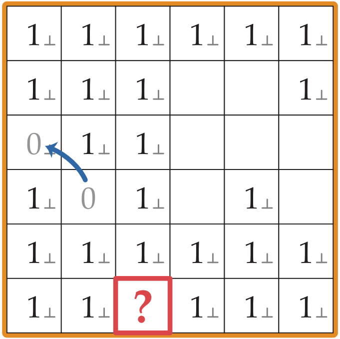

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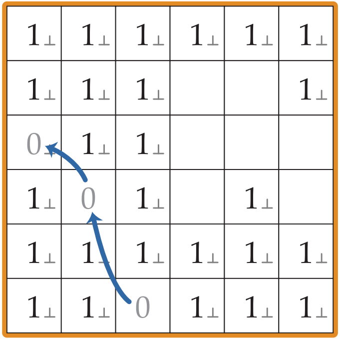

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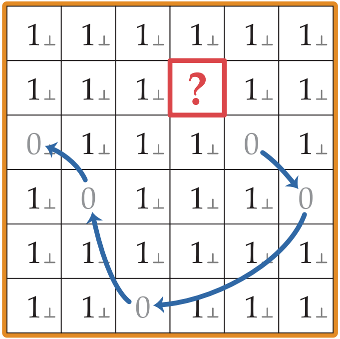

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Other separations inspired by our function

[Ambainis–Balodis–Belovs–Lee–Santha–Smotrovs] (also [Mukhopadhyay–Sanyal]):

- $\mathbf{P}^{\mathbf{dt}}(f) \geq \mathbf{ZPP}^{\mathbf{dt}}(f)^2$
  Counterexample to **Saks–Wigderson'86**!
- $\mathbf{P}^{\mathbf{dt}}(f) \geq \mathbf{BQP}^{\mathbf{dt}}(f)^4$
- $\mathbf{ZPP}^{\mathbf{dt}}(f) \geq \mathbf{RP}^{\mathbf{dt}}(f)^2$

[Ben-David]:

- $\mathbf{BPP}^{\mathbf{dt}}(f) \geq \mathbf{BQP}^{\mathbf{dt}}(f)^{2.5}$

Other query separations

- *Theorem 2:* $$\begin{cases} \mathbf{UP}^{\mathbf{dt}}(f) = k \\ \mathbf{P}^{\mathbf{dt}}(f) = k^2 \end{cases}$$ *(Previous slide)*

$$\Downarrow$$

- *Theorem 1:* $$\begin{cases} \mathbf{2UP}^{\mathbf{dt}}(\mathrm{AND} \circ f^k) = k^2 \\ \mathbf{P}^{\mathbf{dt}}(\mathrm{AND} \circ f^k) = k^3 \end{cases}$$

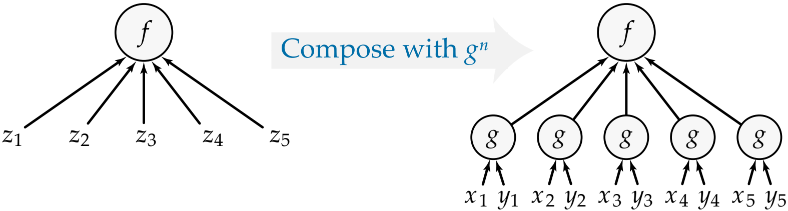
Power 1.5 gap—cf. $\log_3 4 \approx 1.26$ from [Savický'03 / Belovs'06]

- *Theorem 3:* $\exists f: \quad \mathbf{coNP}^{\mathbf{dt}}(f) \geq \Omega(\mathbf{UP}^{\mathbf{dt}}(f)^{1.128})$

$\Longrightarrow$ Involves a delicate recursive construction

Step 2: Simulation theorems

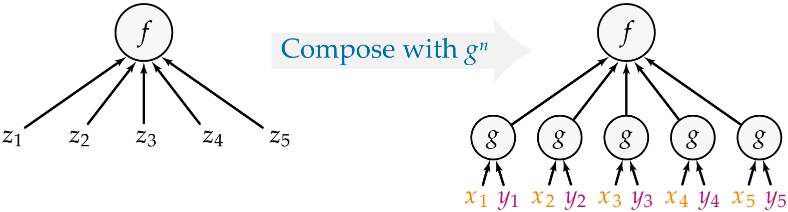
Composed functions $f \circ g^n$



Examples:

- Set-disjointness: $\text{OR} \circ \text{AND}^n$
- Inner-product: $\text{XOR} \circ \text{AND}^n$
- Equality: $\text{AND} \circ \neg\text{XOR}^n$

Composed functions $f \circ g^n$

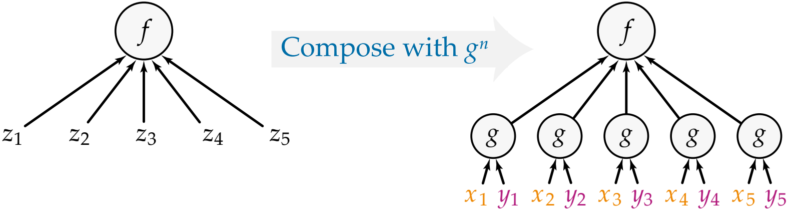


In general: $g\colon \{0,1\}^b \times \{0,1\}^b \to \{0,1\}$ is a small gadget

- **Alice** holds $x \in (\{0,1\}^b)^n$
- **Bob** holds $y \in (\{0,1\}^b)^n$

Inputs x and y encode $z := g^n(x, y)$

Composed functions $f \circ g^n$

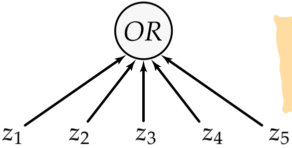


Simulation Theorem Template:

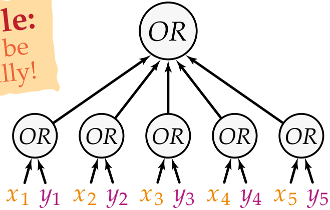
Simulate **cost-C** protocol for $f \circ g^n$ in model $\mathbf{M}^{\mathbf{cc}}$
using **height-C** decision tree for f in model $\mathbf{M}^{\mathbf{dt}}$

i.e., $$\mathbf{M}^{\mathbf{cc}}(f \circ g^n) \approx \mathbf{M}^{\mathbf{dt}}(f \circ g^n)$$

Composed functions $f \circ g^n$



Bad example:
Gadget must be chosen carefully!



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Known simulation theorems

Model	Gadget	Reference
P	$g(x,y) := y_x$ where $\lvert y\rvert = n^{\Theta(1)}$	[Raz–McKenzie, FOCS'97]
NP	$g(x,y) := \langle x,y\rangle \bmod 2$ where $\lvert x\rvert, \lvert y\rvert = \Theta(\log n)$	[GLMZW, STOC'15]
PP	Constant-size g	[Sherstov, STOC'08], [Shi–Zhu, QIC'09]

Simulation for P (Our formulation):

$$\mathbf{P}^{\mathbf{cc}}(f \circ g^n) = \mathbf{P}^{\mathbf{dt}}(f) \cdot \Theta(\log n)$$

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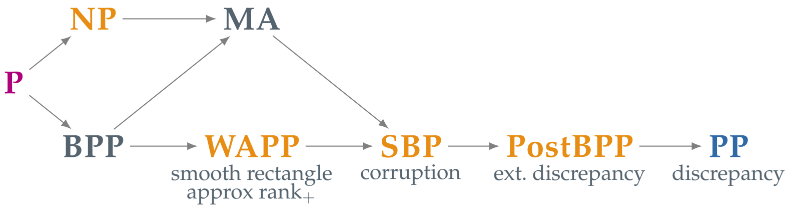

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In progress:

- Find an F with $\mathbf{BPP}^{\mathbf{cc}}(F) \gg \mathbf{2UP}^{\mathbf{cc}}(F)$

  Solved for query complexity:
  [Kothari–Racicot-Desloges–Santha, RANDOM'15]

Open problems:

- Simulation theorem for **BPP**
- Improve gadget size down to $b = O(1)$
  (Gives new proof of $\Omega(n)$ bound for disjointness)

Big challenges:

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- Lower bounds against $\mathbf{PH}^{\mathbf{cc}}$ (or $\mathbf{AM}^{\mathbf{cc}}$)

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Cheers!