Extension Complexity *of* Independent Set Polytopes

Mika Göös — *Harvard University*

Rahul Jain — *National University of Singapore & MajuLab*

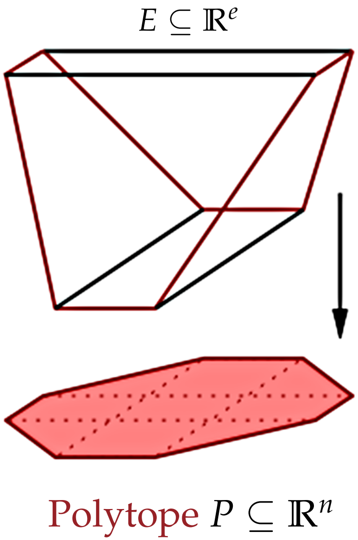
Thomas Watson — *University of Memphis*

Extension complexity

$\mathbf{xc}(P)$

$=$ least # of inequalities in an LP that expresses P

$=$ least # of facets in a polytope $E \subseteq \mathbb{R}^e$ that projects onto P



The P-versus-NP page

This page collects links around papers that try to settle the "P versus NP" question (in either way). Here are some links that explain/discuss this question:

Milestones

Note: The following paragraphs list many papers that try to contribute to the P-versus-NP question. Among all these papers, there is only **a single paper** that has appeared in a peer-reviewed journal, that has thoroughly been verified by the experts in the area, and whose correctness is accepted by the general research community: The paper by Mihalis Yannakakis. (And this paper does not settle the P-versus-NP question, but "just" shows that a certain approach to settling this question will never work out.)

1. **[Equal]:** In 1986/87 Ted Swart (University of Guelph) wrote a number of papers (some of them had the title: *"P=NP"*) that gave linear programming formulations of polynomial size for the Hamiltonian cycle problem. Since linear programming is polynomially solvable and Hamiltonian cycle is NP-hard, Swart deduced that P=NP.
In 1988, Mihalis Yannakakis closed the discussion with his paper "Expressing combinatorial optimization problems by linear programs" (Proceedings of STOC 1988, pp. 223-228). Yannakakis proved that expressing the traveling salesman problem by a symmetric linear program (as in Swart's approach) requires exponential size. The journal version of this paper has been published in Journal of Computer and System Sciences 43, 1991, pp. 441-466.

2. **[Equal]:** The 1996 issue (Volume 1, 1996, pp. 16-29) of the "SouthWest Journal of Pure and Applied Mathematics" (SWJPAM) contains the article *"Polynomial-Time Partition of a Graph into Cliques"* by the Ukrainian mathematician Anatoly Plotnikov. This article designs a polynomial time algorithm for an NP-hard graph problem, and thus proves P=NP.
(SWJPAM is an electronic journal devoted to all aspects of Pure and Applied mathematics, and related topics. Authoritative expository and survey articles on subjects of special interest are also welcomed. SWJPAM serves as an international forum for the publication of high-quality strictly peer-reviewed original research articles. The article is usually sent to at least two experts in the area. Two positive reviews are required for the acceptance and publication of any submitted article.)

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First explicit lower bounds by

Fiorini–Massar–Pokutta–Tiwary–de Wolf

STOC 2012

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Our result

Main result

There are n-node graphs G whose independent set polytope P_G satisfies $\mathrm{xc}(P_G) \geq 2^{\Omega(n/\log n)}$

Existential: Most n-dimensional $0/1$-polytopes have $\mathrm{xc}(P) = 2^{\Theta(n)}$ [Rot12]

Explicit: Previous best $2^{\Omega(\sqrt{n})}$ for **Cut**, **TSP**, and **Matching** polytopes [FMP+12, Rot13]

Upper bounds: Planar, bounded treewidth, . . . [many]

Key conceptual idea:

KW/EF connection

[Hrubeš–Razborov]

KW/EF: Overview

KW

[Raz–Wigderson'90]:
Matching function ($\binom{m}{2}$ input bits) has monotone formula complexity $2^{\Omega(m)}$

EF

[Rothvoß'13]:
Perfect matching polytope has extension complexity $2^{\Omega(m)}$

Monday, December 30, 2013

2013 Complexity Year in Review

The complexity result of the year goes to The Matching Polytope has Exponential Extension Complexity by Thomas Rothvoss. Last year's paper of the year showed that the Traveling Salesman Problem cannot have a subexponential-size linear program formulation. If one could show that every problem in P has a short polynomial-size LP formulation then we would have a separation of P and NP. Rothvoss' paper shoots down that approach by giving an exponential lower bound for the polynomial-time computable matching problem. This story is reminiscent of the exponential monotone circuit lower bounds first for clique then matching in the 1980's.

KW/EF: Overview

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[Göös–Pitassi'14]:
Explicit n-bit function with mon. formula complexity $2^{\Omega(n/\log n)}$

This work:
Explicit n-dim. $0/1$-polytope of extension complexity $2^{\Omega(n/\log n)}$

Definitions

KW$^+$-game

Input: **Alice** gets $x \in f^{-1}(1)$, **Bob** gets $y \in f^{-1}(0)$

Output: An index $i \in [n]$ such that $x_i = 1$ and $y_i = 0$

$$\begin{aligned} \log \text{ mon. formula size of } f &= \text{mon. circuit depth of } f \\ &= \text{det. cc of KW}^+\text{-game of } f \end{aligned}$$

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EF

[Yannakakis'89]:
- $\mathrm{xc}(P) =$ nonneg. rank of **slack matrix**
- Suppose $P = \{x \in \mathbb{R}^n : Ax \geq b\}$
- The (facet f, vertex v)-entry of slack matrix is $A_f v - b_f \geq 0$

[Faenza et al.'11]: log nonneg. rank of M

$=$ randomised cc of accepting input (x, y) with probability $\propto M_{xy}$

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EF

- Define $F := \text{conv}\, f^{-1}(1)$
- Express the existence of a solution to the KW$^+$-game:

$$\sum_{i:y_i=0} x_i \geq 1 \qquad \text{with slack} \qquad \sum_{i:y_i=0} x_i - 1$$

- Valid for $x \in F$ and $y \in f^{-1}(0)$. Hence get a **submatrix** of the slack matrix of F.

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(#∃−1)-game

Input: **Alice** gets $x \in f^{-1}(1)$, **Bob** gets $y \in f^{-1}(0)$

Output: Accept with probability proportional to
# of witnesses minus one

(#∃−1)-game for $f \;\leq\; \log \mathrm{xc}(F)$

Definitions

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Input: **Alice** gets $x \in f^{-1}(1)$, **Bob** gets $y \in f^{-1}(0)$

Output: An index $i \in [n]$ such that $x_i = 1$ and $y_i = 0$

$(\#\exists{-}1)$-game

Input: **Alice** gets $x \in f^{-1}(1)$, **Bob** gets $y \in f^{-1}(0)$

Output: Accept with probability proportional to
\# of witnesses minus one

KW/EF connection:

$(\#\exists{-}1)$-game for $f \;\leq\; O(\mathrm{KW}^+(f))$

The connection — Hrubeš–Razborov

KW/EF connection:

$$(\#\exists{-}1)\text{-game for } f \;\leq\; O(\mathrm{KW}^+(f))$$

Proof:

1. Find particular witness $i^* \in [n]$ ($x_{i^*} = 1, y_{i^*} = 0$); uses $\mathrm{KW}^+(f)$ bits
2. Sample random $i \in [n] - i^*$ and accept iff i is a witness; uses $\log n \leq \mathrm{KW}^+(f)$ bits

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Compare with [RPRC'16] from yesterday:

"Exponential Lower Bounds for Monotone Span Programs"

This is nonnegative analogue of Razborov's rank method!

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Proof strategy:

Query-to-communication lifting

(theme of my PhD thesis)

9:15 a.m. – 10:00 a.m. **Toniann Pitassi**
The amazing power of composition

- **3:40-4:30** Justin Thaler

Chebyshev Polynomials, Approximate Degree, and Their Applications.

3:20 - 3:40 **On the Communication Complexity of Approximate Fixed Points**
Tim Roughgarden, Omri Weinstein

4:40 - 5:00 **Settling the Complexity of Computing Approximate Two-player Nash Equilibria**
Aviad Rubinstein

9:00 - 9:20 **Structure of Protocols for XOR Functions**
Hamed Hatami, Kaave Hosseini, Shachar Lovett

2:55 - 3:15 **Exponential Lower Bounds for Monotone Span Programs**
Robert Robere, Toniann Pitassi, Benjamin Rossman, Stephen A. Cook

9:50 - 10:10 **How Limited Interaction Hinders Real Communication (and What It Means for Proof and Circuit Complexity)**
Susanna F. de Rezende, Jakob Nordström, Marc Vinyals

9:50 - 10:10 **Separations in Communication Complexity using Cheat Sheets and Information Complexity**
Anurag Anshu, Aleksandrs Belovs, Shalev Ben-David, Mika Göös, Robin Kothari, Rahul Jain, Troy Lee, Miklos Santha

[2] arXiv:1610.02704 [pdf, other]
Approximating Rectangles by Juntas and Weakly-Exponential Lower Bounds for LP Relaxations of CSPs
Pravesh Kothari, Raghu Meka, Prasad Raghavendra

Idealised Strategy

Query world:

1. Start with TSEITIN **query** search problem (has $O(1)$-bit certificates)
2. Prove that $(\#\exists-1)$-game for TSEITIN has query complexity $\Omega(n)$

Query to communication:

3. Lift into **communication** search problem TSEITIN $\circ\, g$ for a small g
   - Lifting theorem $\Longrightarrow$ $(\#\exists-1)$-game for TSEITIN $\circ\, g$ is hard

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   - Lifting theorem $\implies$ $(\#\exists{-}1)$-game for $\text{TSEITIN} \circ g$ is hard

Communication world:

4. Embed $\text{TSEITIN} \circ g$ inside the KW^+-game for $f :=$ CSP-SAT (monotone variant)
5. Reduce $F = \operatorname{conv} f^{-1}(1)$ to an independent set polytope

More fun with KW/EF

Further observations:

1. KW/EF connection fails for **monotone circuit size**
2. Explicit lower bounds for the independent set polytopes of "sparse paving" matroids would imply ***non***-monotone circuit lower bounds

Open problems to attack via KW/EF?

1. Extension complexity of **matroid** independent set polytopes
2. Separate **LP** and **SDP** extension complexities for polytopes
   Related: Separate real mon. span programs and mon. formulas
   (log-rank conjecture for search problems)

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Cheers!