Lower Bounds *for* Local Approximation

Mika Göös, Juho Hirvonen & Jukka Suomela *(HIIT)*

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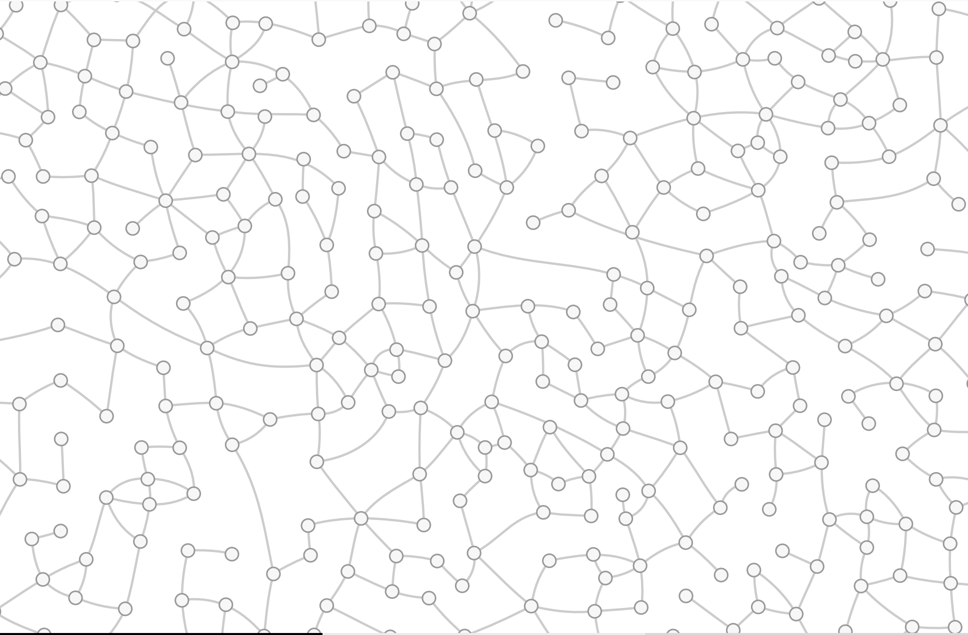
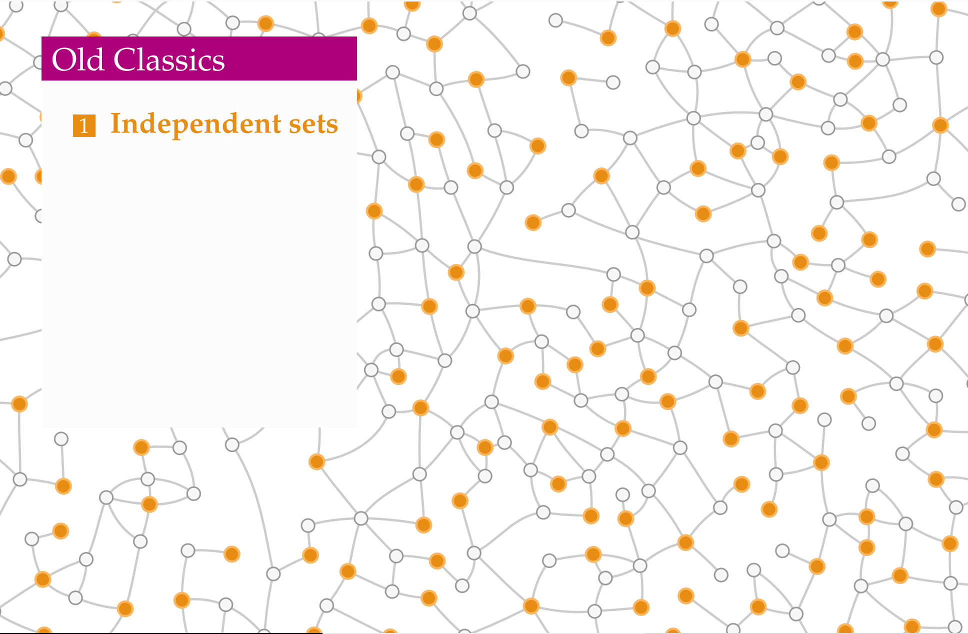
We prove: **Local algorithms** do not need ***unique IDs*** when computing **approximations** to graph optimization problems

Input = Graph $\mathcal{G}$ = Communication Network



“Simple” Graph Problems



Old Classics

1. **Independent sets**

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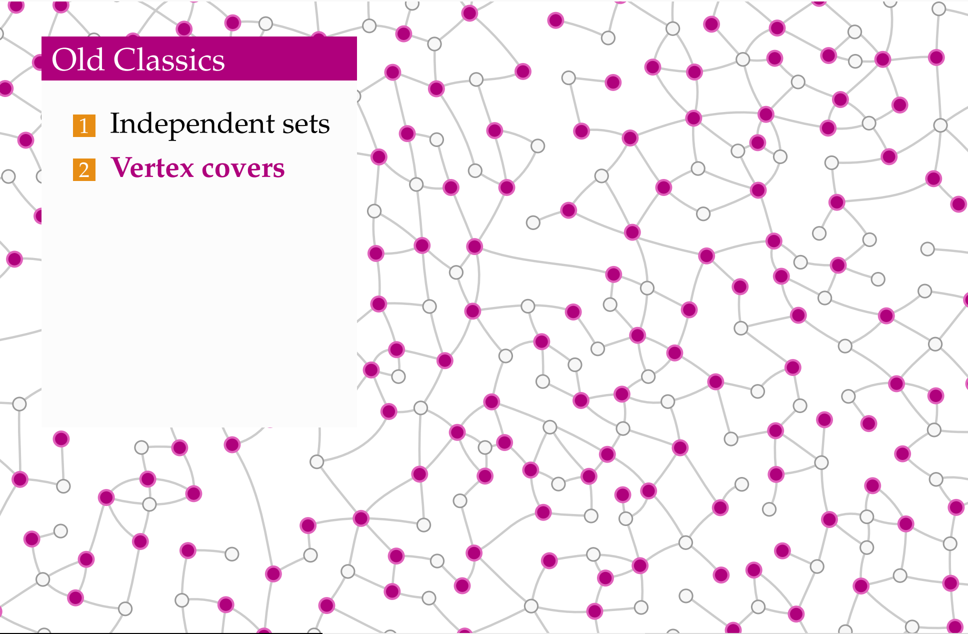

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1. Independent sets
2. **Vertex covers**

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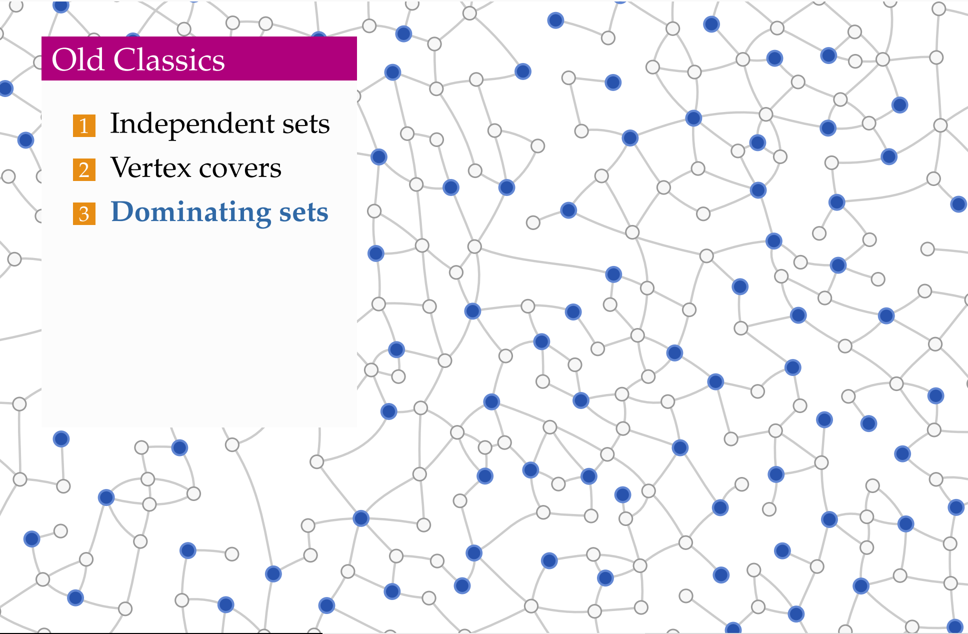

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Old Classics

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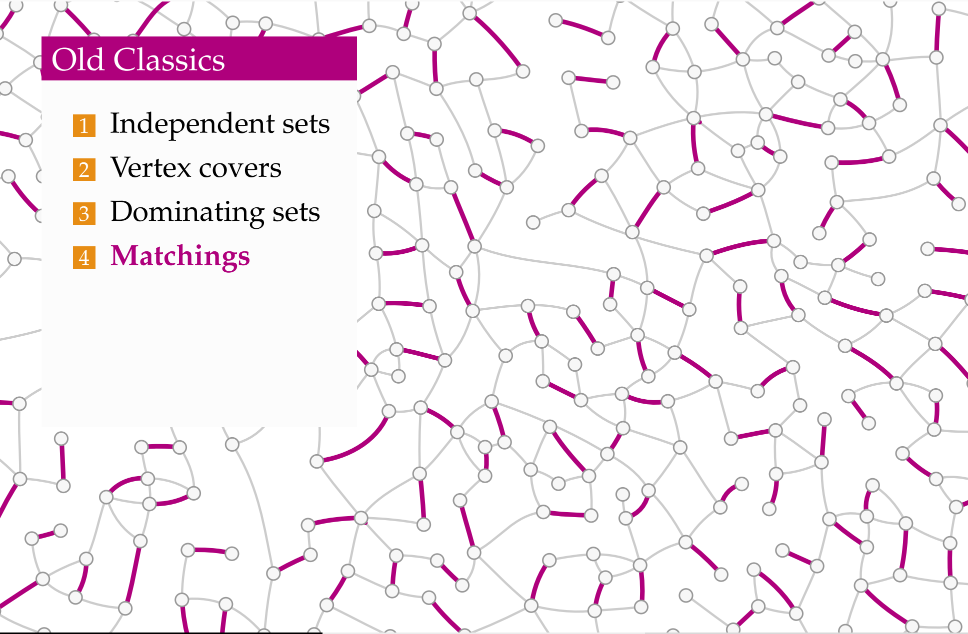
4. **Matchings**



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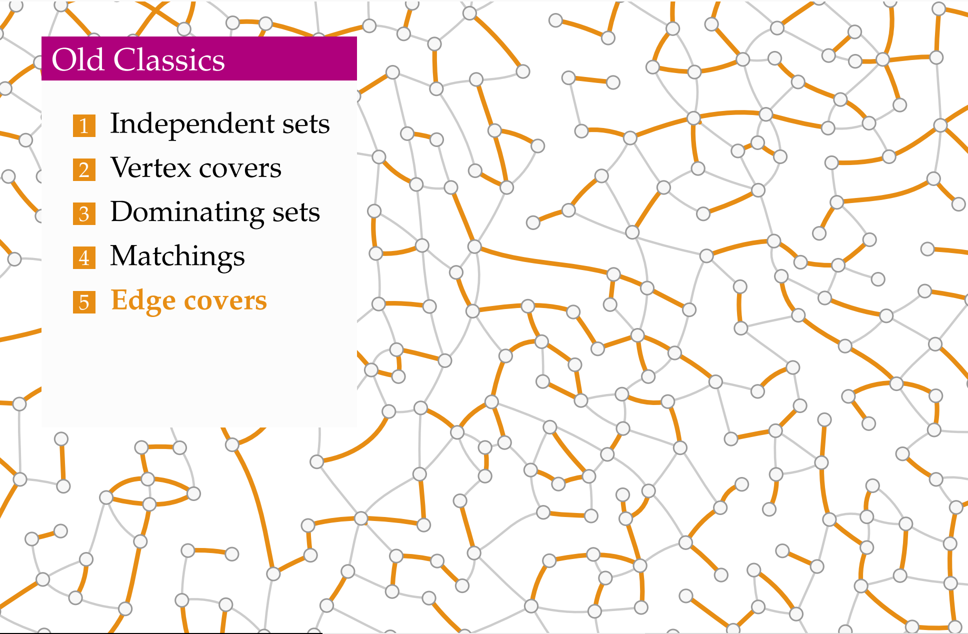

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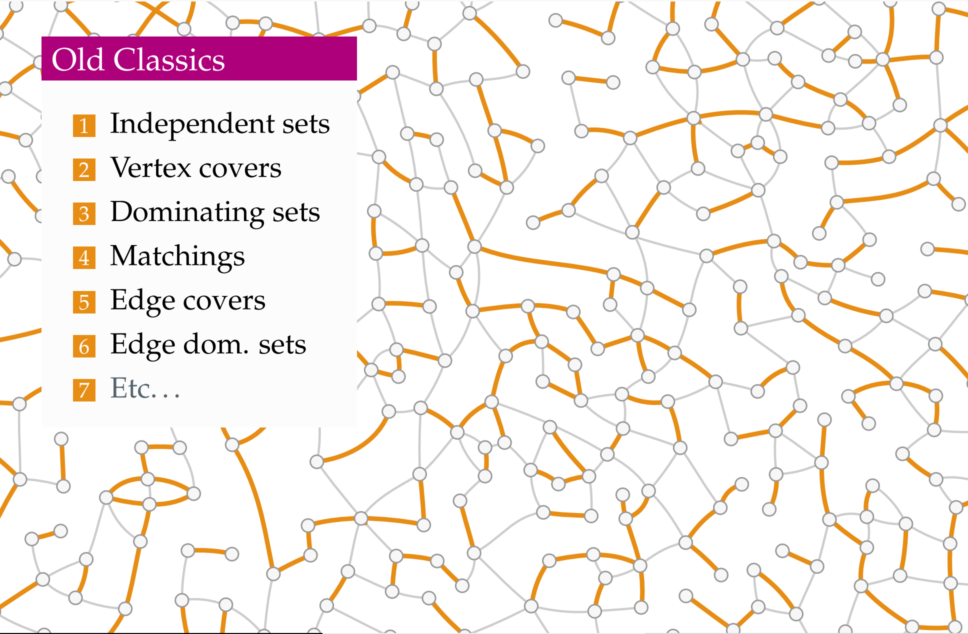

Old Classics

1. Independent sets
2. Vertex covers
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5. Edge covers
6. Edge dom. sets
7. Etc. . .

Local Algorithms

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2. Deterministic, synchronous

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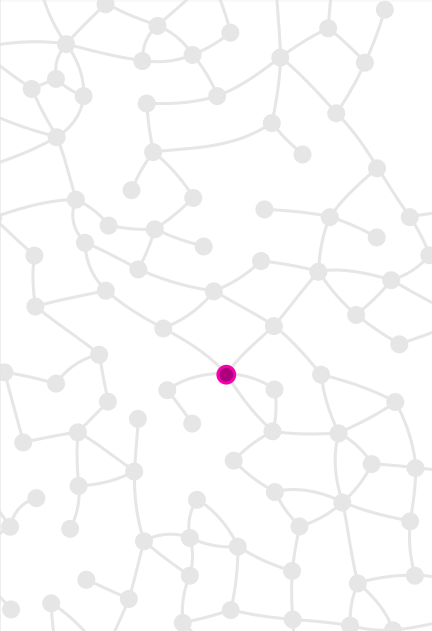

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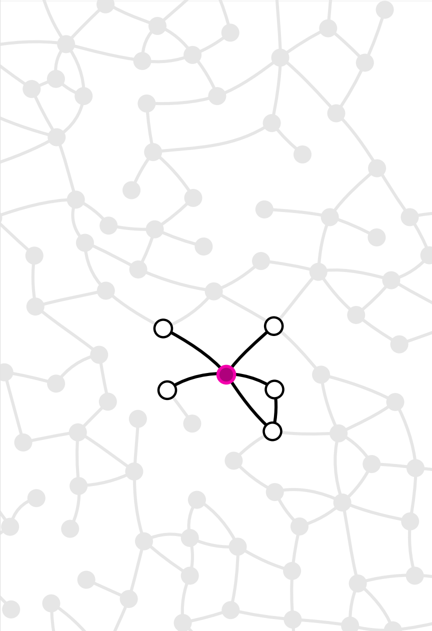

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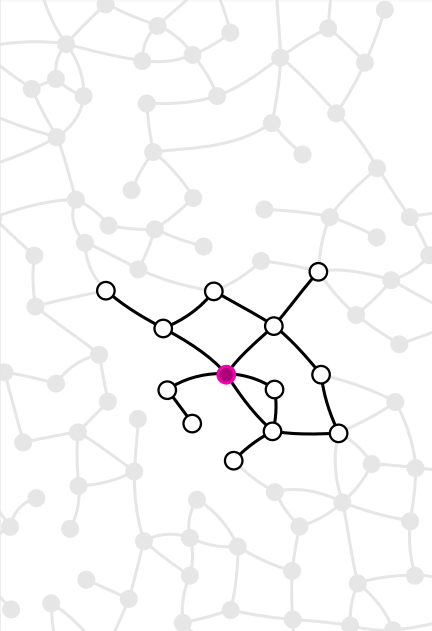

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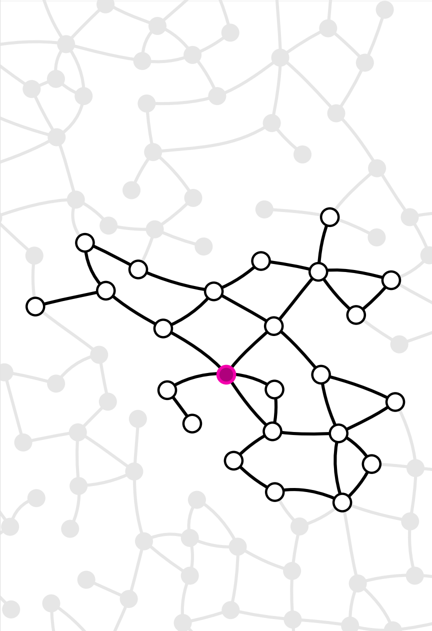

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3. **Locality:** running time r of **A** is
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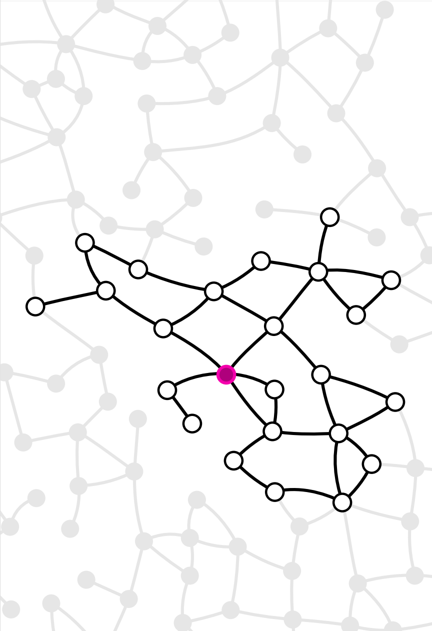

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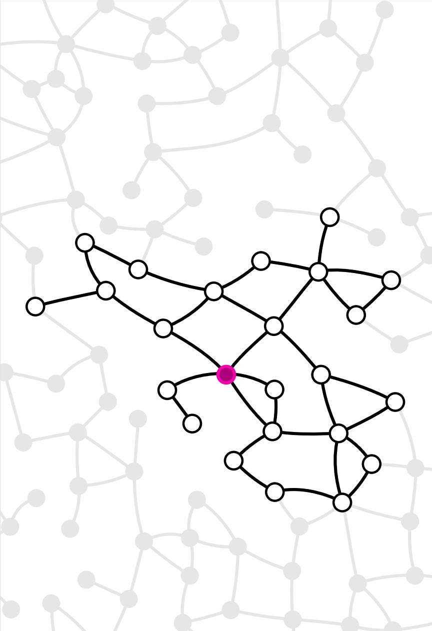
On ***bounded degree graphs*** ($\Delta = O(1)$) running time is a constant:

$$r \in \mathbb{N} \qquad \text{(e.g., } r = \mathbf{3}\text{)}$$



Local Algorithms

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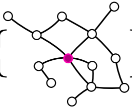
$\mathbf{A} : \{$  $\} \to \{0, 1\}$

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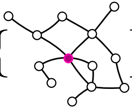
$\mathbf{A} : \{$  $\} \to \{0, 1\}$

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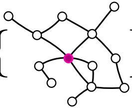
$\mathbf{A} : \{$  $\} \to \{0,1\}^{\Delta}$

Output:

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Edge set: $\mathbf{A}(\mathcal{G}, v)$ is a vector of length Δ indicating which edges incident to v are included

Two Network Models

Unique Identifiers

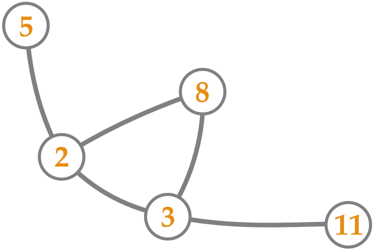
Anonymous Networks
with Port Numbering

Two Network Models

Unique Identifiers

- Each node has a unique $O(\log n)$-bit **label**:

$$V(\mathcal{G}) \subseteq \{1, 2, \ldots, \mathrm{poly}(n)\}$$



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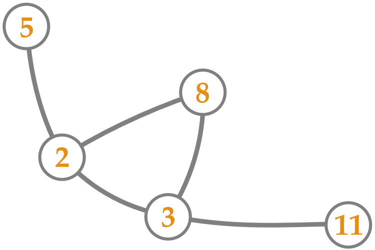
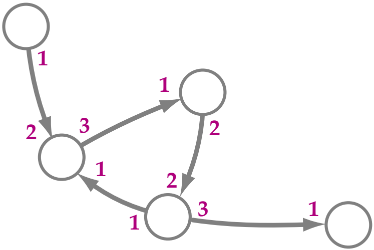


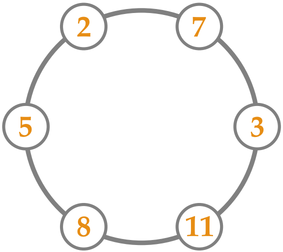
Anonymous Networks
with Port Numbering

- Node v can refer to its neighbours via **ports** $1, 2, \dots, \deg(v)$
- Edges are **oriented**



Example: Independent Sets on a Cycle

ID-model



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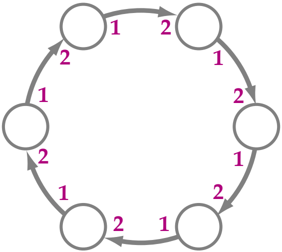

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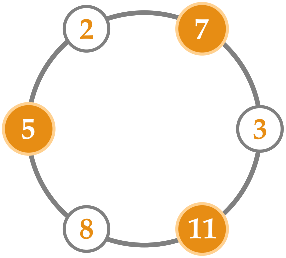

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  Maximal independent set
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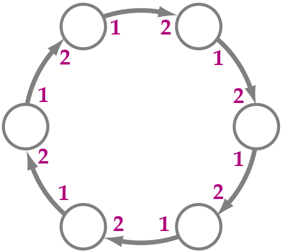

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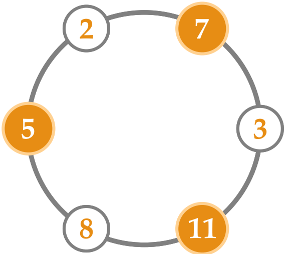

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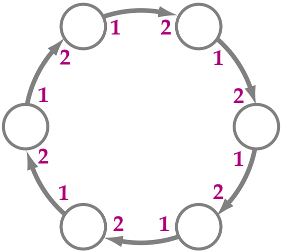

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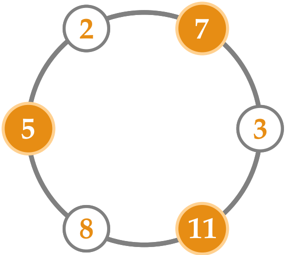

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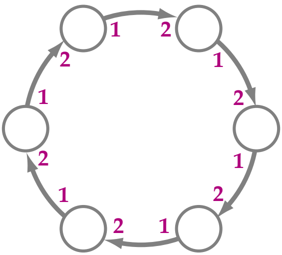

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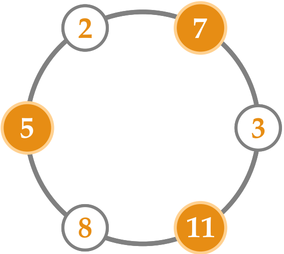

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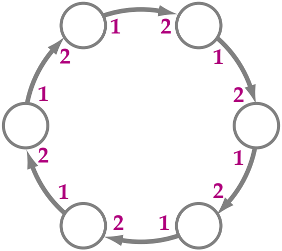

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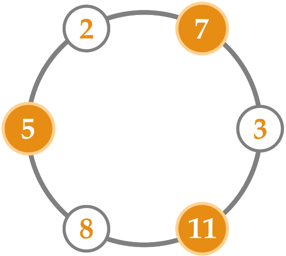
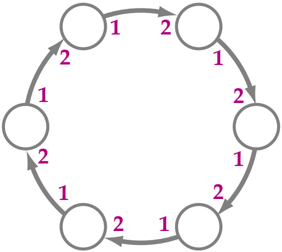
ID-model $\gg$ **in general** **PO-model**





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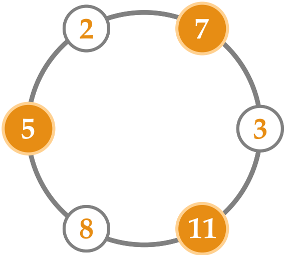

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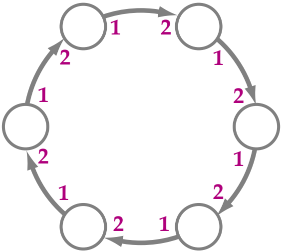

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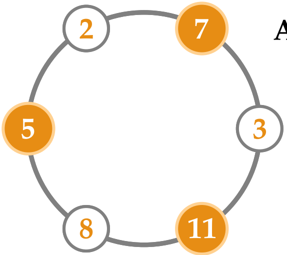

$\approx$ **for Local Approximation**

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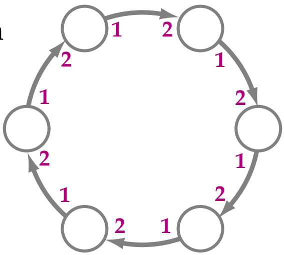

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Known Approximation Ratios

	ID	PO
Max **Independent Set**	∞	∞
Max **Matching**	∞	∞
Min **Vertex Cover**	2	2
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Min **Dominating Set**	$\Delta' + 1$	$\Delta' + 1$
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Min **Edge Dominating Set**	α	$4 - 2/\Delta'$

$$3 \le \alpha \le 4 - 2/\Delta' \quad ???$$

Our Result (informally)

Main Thm: When **Local Algorithms** compute **constant factor** approximations,

$$\mathbf{ID} = \mathbf{PO}$$

for a general class of graph problems

Proof!

OI: Order Invariant Algorithms

[Naor–Stockmeyer 95]:

Ramsey's theorem implies that **local ID-algorithms** can only **compare** identifiers

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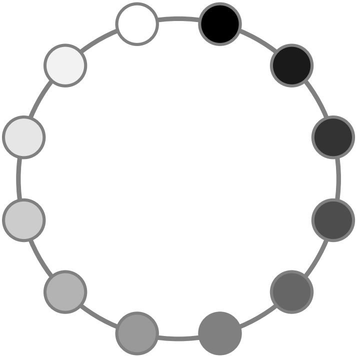
Order invariant algorithm A:

Input: Ordered graph $(\mathcal{G}, \leq)$

Output: $\mathbf{A}(\mathcal{G}, \leq, v)$ depends only on **order type** of the radius-r neighbourhood of v

Example: Ordered Cycle



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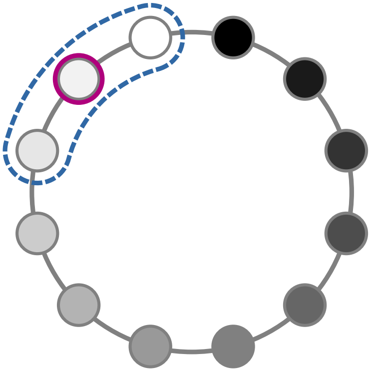

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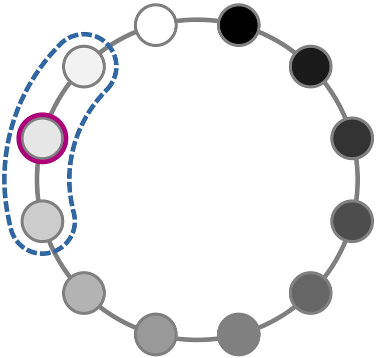

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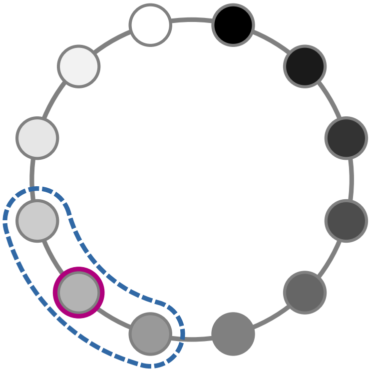

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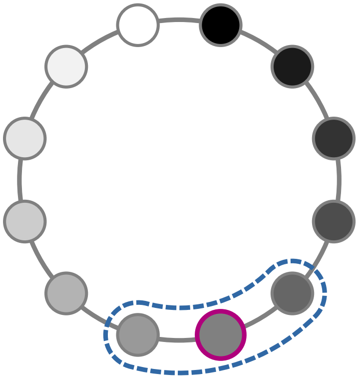

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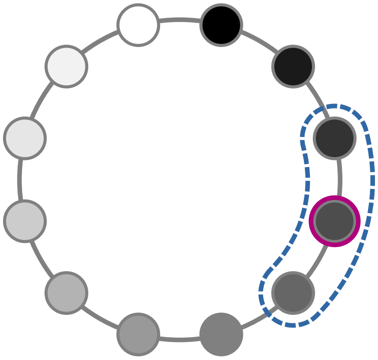

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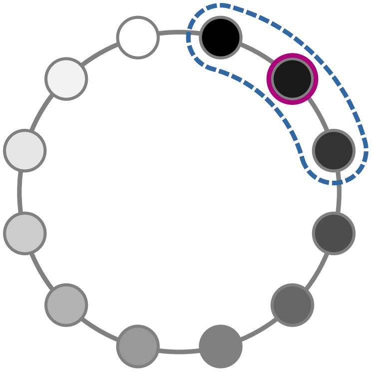

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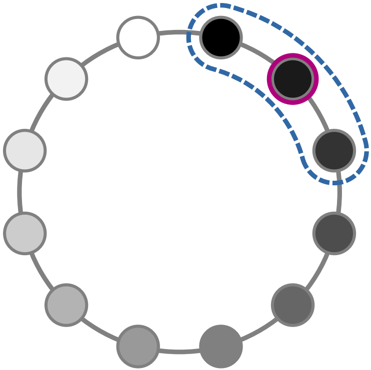

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  $(1-\epsilon)$**-fraction**

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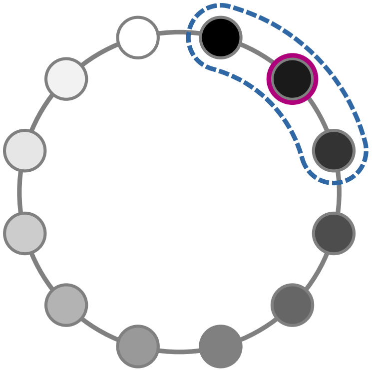

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- **OI-algorithm** outputs the same almost everywhere!

Two Advantages of OI over PO

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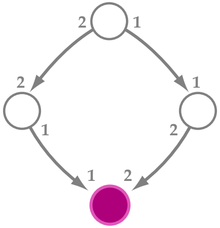

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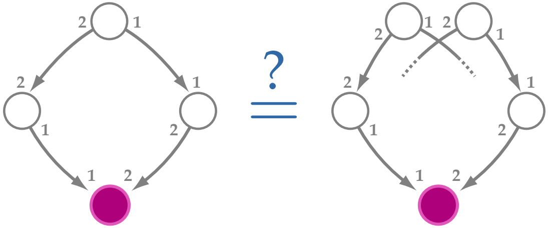

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- **We assume:** Feasibility of a solution can be checked by a **PO**-algorithm

Two Advantages of OI over PO

$$\mathbf{1 + 2 =}$$

Need to understand

linear orders on **high girth graphs**

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Definition: We say that an **ordered** graph $(\mathcal{G}, \leq)$ is

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Example: **Large cycles** are $(1 - \epsilon, r)$-homogeneous ($\alpha = 1$ not possible)

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1. $(1 - \epsilon, r)$-homogeneous
2. $2k$-regular
3. Large girth
4. Finite graph

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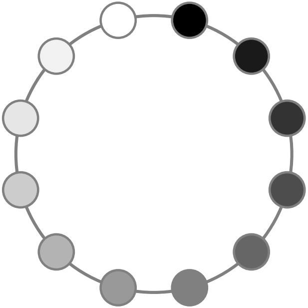

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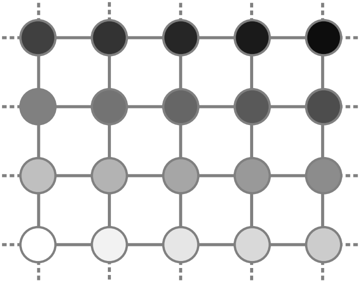

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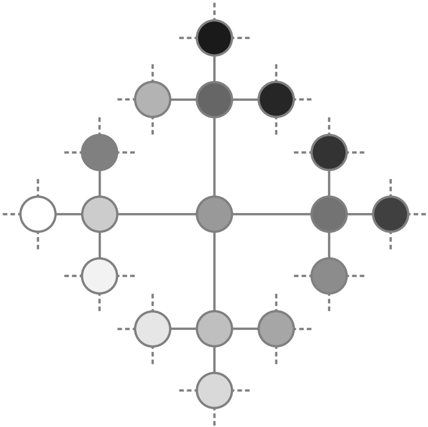

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Main Technical Result:

Graphs $(\mathcal{H}_\epsilon, \leq_\epsilon)$ with properties 1–4 exist

Homogeneous High Girth Graphs

1. $(1-\epsilon, r)$-homogeneous
2. $2k$-regular
3. Large girth
4. Finite graph

Main Technical Result:

Graphs $(\mathcal{H}_\epsilon, \leq_\epsilon)$ with properties 1–4 exist

Proof. We use **Cayley graphs** of **soluble groups**

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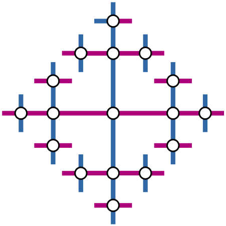
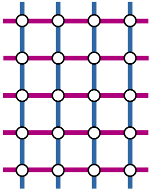


$F(\alpha, \beta)$

?



$\mathbb{Z} \times \mathbb{Z}$

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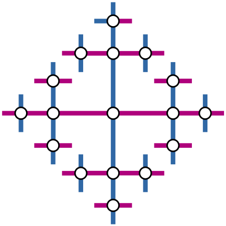
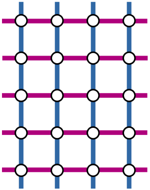


$F(\alpha, \beta)$

Want:

- **Invariant order:**
  $x < y \iff gx < gy$



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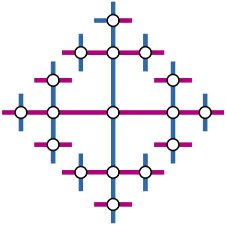
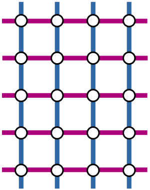


$F(\alpha, \beta)$

Want:

- Invariant order
- **Polynomial growth** [Gromov's theorem]



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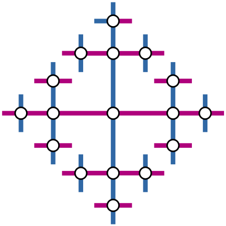
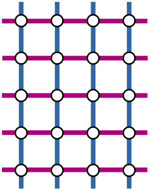


$F(\alpha, \beta)$

Want:

- Invariant order
- Polynomial growth
- **Large girth** [Gamburd et al. 09]



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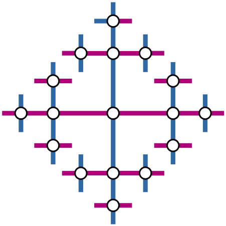

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Want:

- Invariant order
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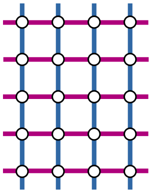
- Large girth

$$G_1 = \mathbb{Z},$$
$$G_{i+1} = (G_i \times G_i) \rtimes \mathbb{Z}$$



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Proof of Main Thm:

Form graph products $(\mathcal{H}_\epsilon, \leq_\epsilon) \times \mathcal{G}$

Conclusion

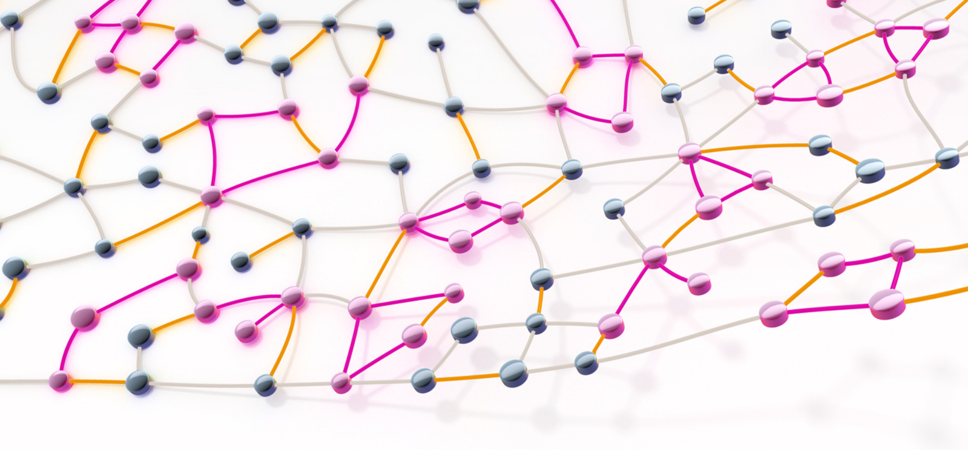
Our result:

For Local Approximation,

$$\mathbf{ID} = \mathbf{OI} = \mathbf{PO}$$

Open problems:

- Planar graphs?
- Applications elsewhere—descriptive complexity?



Cheers!