# Adventures in Monotone Complexity and **TFNP**


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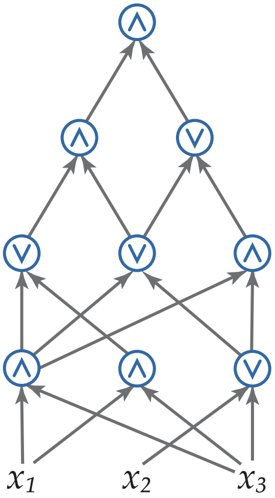
# Monotone complexity

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**Monotone:**

$x \leq y \implies f(x) \leq f(y)$

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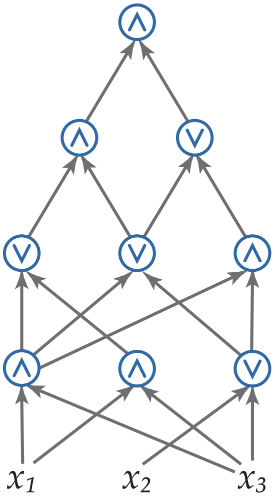


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**Monotone:**

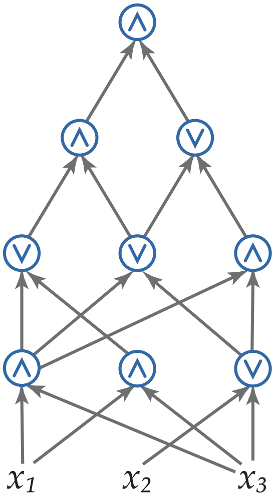
$x \leq y \implies f(x) \leq f(y)$

[Razborov'85]:

$k$-Clique: $\{0,1\}^{\binom{n}{2}} \to \{0,1\}$
has no small monotone circuits

$\implies$ **P** $\neq$ **NP** in monotone world

# Monotone complexity

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**Monotone:**

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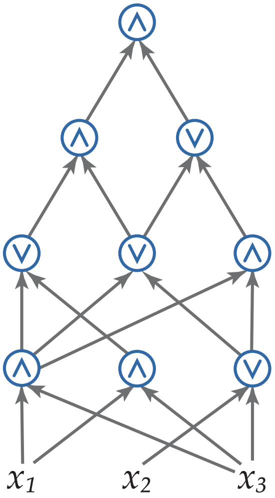
[Tardos'88]:

$\textsc{Tardos} \in \mathbf{P}$

has no small monotone circuits

# Monotone complexity




## Connections:

1. Communication complexity
   *KW games; communication* **TFNP**
2. Proof complexity
   *Monotone interpolation*
3. Extended formulations
   *Hrubeš–Razborov*
4. Cryptography
   *Secret sharing*

# Separation result

**Main Theorem**

Mon. circuit complexity of $\textsc{Xor-Sat}_n$ is $2^{n^{\Omega(1)}}$

$$\begin{aligned} v_1 \oplus v_2 \oplus v_3 &= 1 \\ v_1 \oplus v_2 \oplus v_3 &= 0 \\ &\vdots \\ v_{n-2} \oplus v_{n-1} \oplus v_n &= 1 \end{aligned}$$

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$$
\begin{array}{ccc}
\text{input } x & & \\
\downarrow & & \\
\mathbf{1} & v_1 \oplus v_2 \oplus v_3 & = 1 \\
\mathbf{0} & v_1 \oplus v_2 \oplus v_3 & = 0 \\
\vdots & \vdots & \\
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$\textsc{Xor-Sat}_n(x) = 1 \iff x$ is unsatisfiable

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**Monotone vs. non-monotone separations:**

- $\textsc{Xor-Sat} \in \mathbf{NC}^2$
- $\textsc{Tardos} \in \mathbf{P}$ [Tardos'88]
- $\textsc{Matching} \in \mathbf{RNC}^2$ [Razborov'85]

# Example monotone computation

$v_1 \oplus v_2 = 0$ $\qquad v_3 = 1$ $\qquad v_1 \oplus v_3 = 1$

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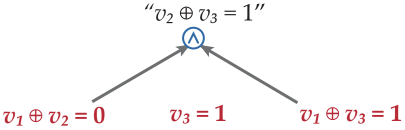

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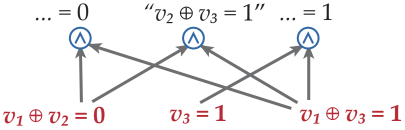

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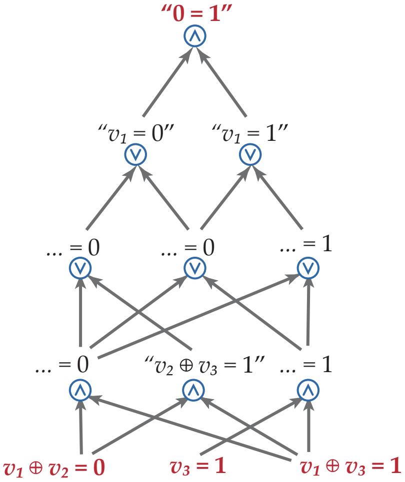

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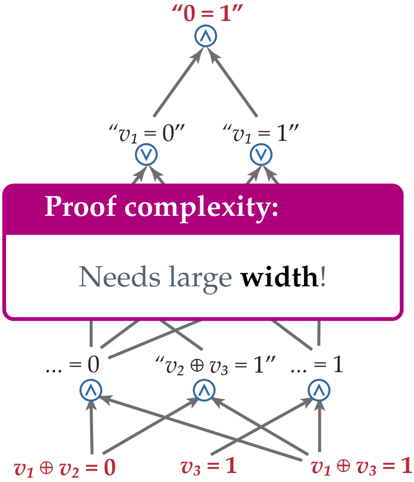


**Proof complexity:**

Needs large **width**!

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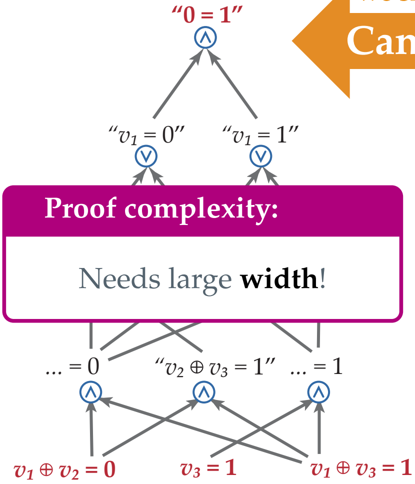


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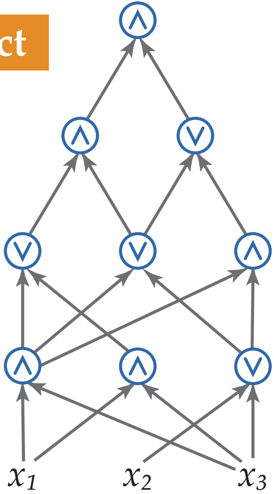
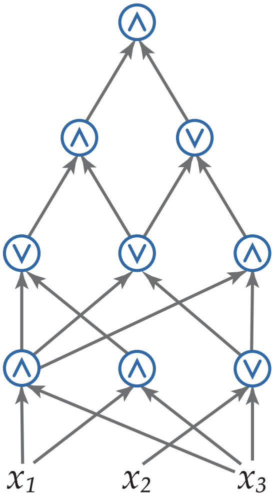
we show:

**Can extract**



## Circuits $\leftrightarrow$ Communication protocols

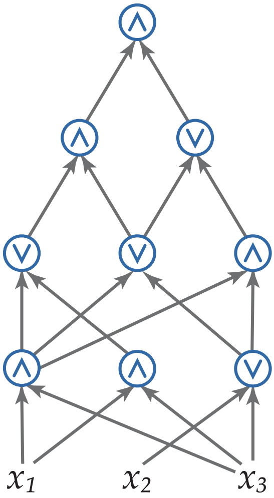
*Karchmer–Wigderson, 1988*
*Razborov, 1995*




# Monotone KW game

for monotone $f\colon \{0,1\}^n \to \{0,1\}$

| | |
|---|---|
| **Alice** | $x \in f^{-1}(1)$ |
| **Bob** | $y \in f^{-1}(0)$ |
| Output | $i$ with $x_i > y_i$ |

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$=$ CC of **mKW game**

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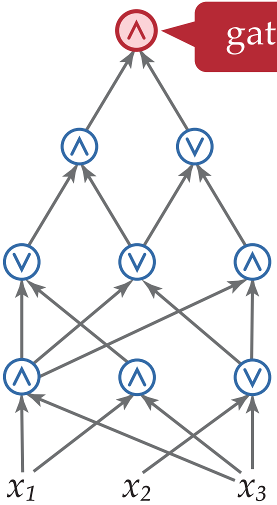


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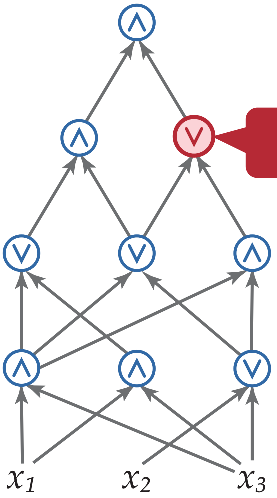


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gate($x$) > gate($y$)

$^{-1}(1)$

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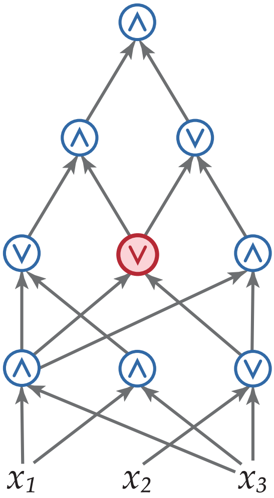


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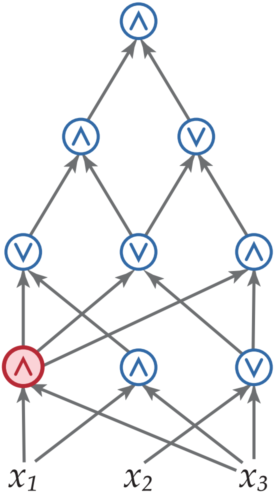


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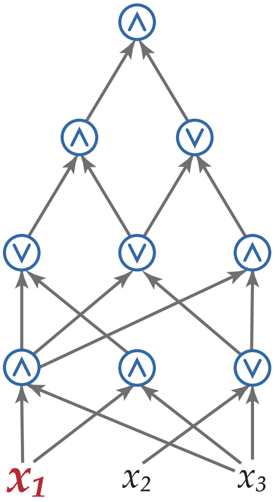


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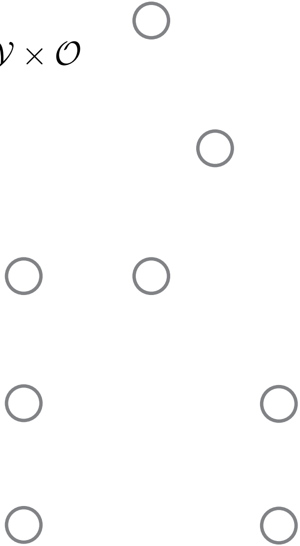
Defined by $(V, \Pi_v)$ where for $v \in V$



# Circuit size [R'95]

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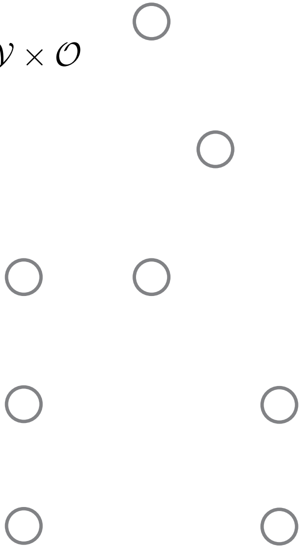

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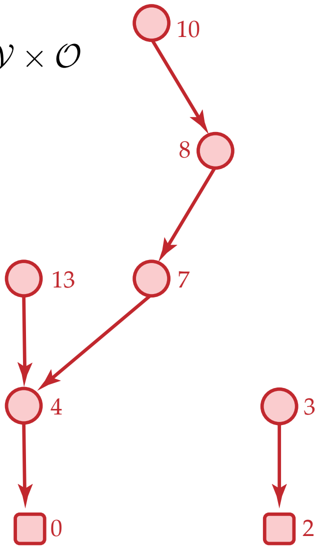

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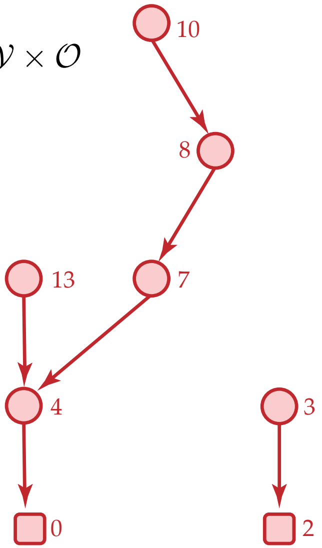

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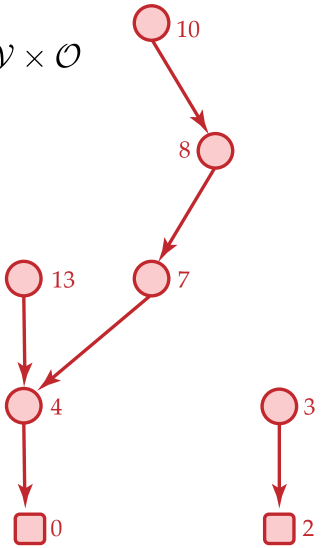

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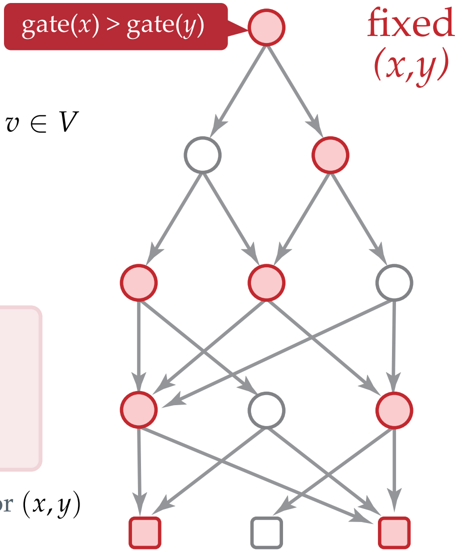

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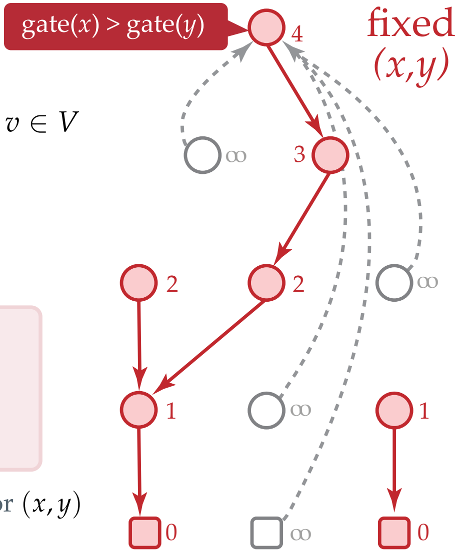

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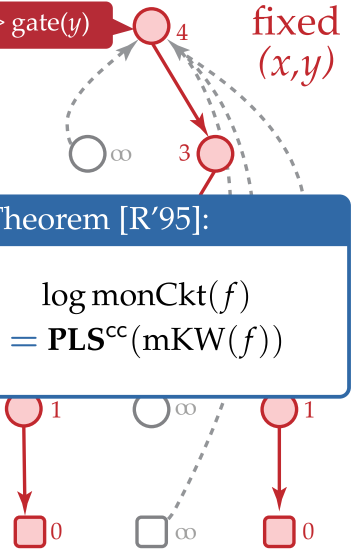


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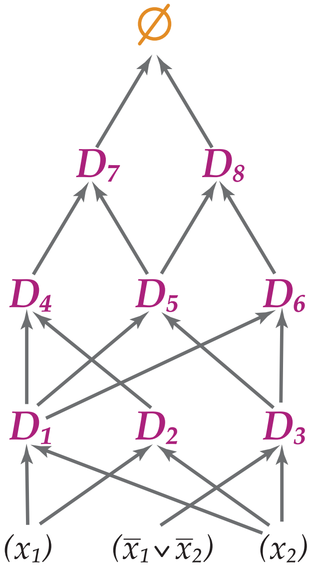
Query analogue of **PLS**:

*Resolution proof system*

# PLS$^{\mathrm{dt}}$ and Resolution

Unsat $k$-CNF formula $F$



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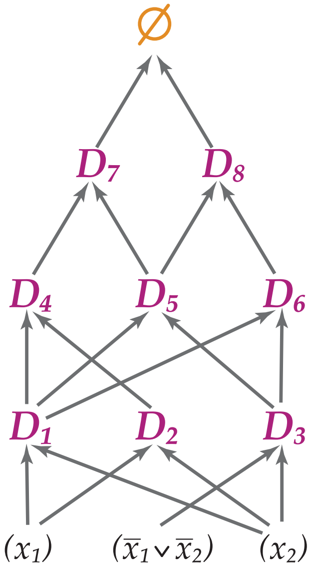

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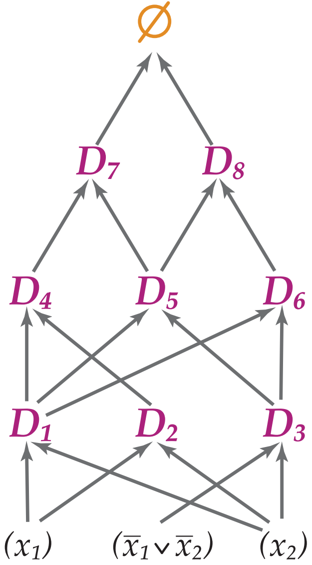
Same as **PLS**$^{\text{cc}}$-protocol except:

- Vertices have decision trees $\mathcal{T}_v$
- **Cost:** max height of $\mathcal{T}_v$



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Resolution width of $F$
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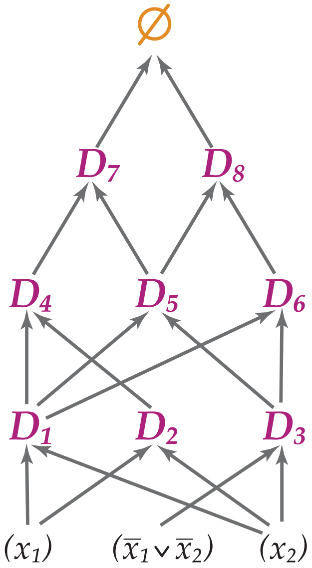

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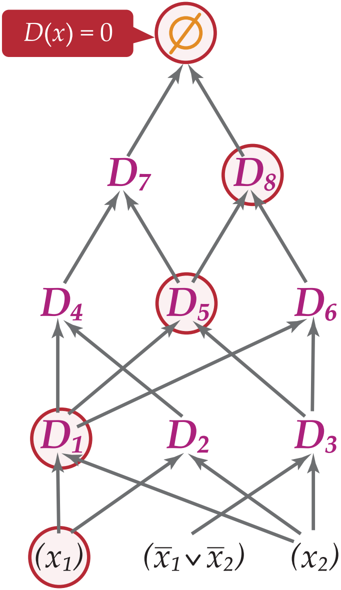

# Query-to-communication lifting

$$\mathbf{PLS}^{\mathsf{cc}}(S(F) \circ g) \quad \overset{\text{[GGKS'18]}}{=} \quad \mathbf{PLS}^{\mathsf{dt}}(S(F))$$

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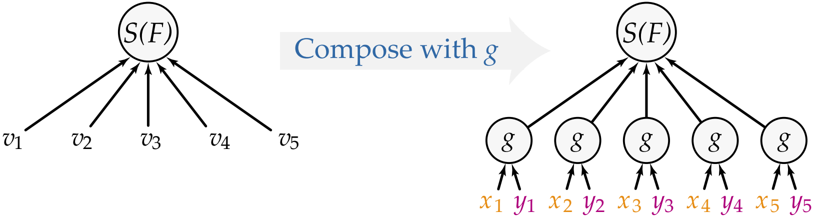


*Index gadget* $g\colon [m]\times\{0,1\}^m \to \{0,1\}$ mapping $(x,y)\mapsto y_x$

# Query-to-communication lifting

$$\mathbf{PLS}^{\mathsf{cc}}(S(F)\circ g) \overset{\text{[GGKS'18]}}{=} \mathbf{PLS}^{\mathsf{dt}}(S(F))$$

$$\mathbf{PLS}^{\mathsf{cc}}(S(F)\circ g) \geq \mathbf{PLS}^{\mathsf{cc}}(\mathrm{mKW}(f)) = \log \mathrm{monCkt}(f)$$

$$\mathbf{PLS}^{\mathsf{dt}}(S(F)) = \text{Resolution width of } F$$

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$$\downarrow$$

XOR-SAT is hard

$$\mathbf{PLS}^{\mathsf{dt}}(S(F)) = \text{Resolution width of } F$$

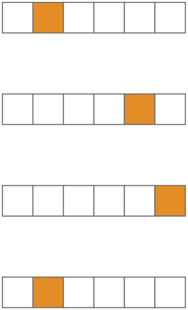
$$\uparrow$$

XOR-CSP $F$

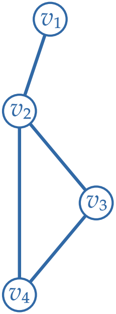
# Reduction: $S(F) \circ g \leq \mathrm{mKW}(\textsc{Xor-Sat})$

## Alice

$x \in [m]^4$



unsat $F$




## Bob

$y \in \{0,1\}^{4m}$

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 |

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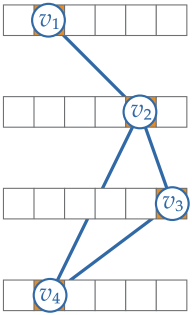
**Alice**

$x \in [m]^4$




$\in \textsc{Xor-Sat}^{-1}(1)$

unsat $F$

**Bob**

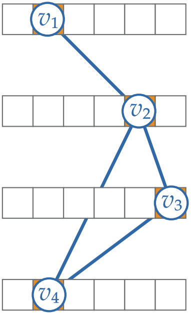
$y \in \{0,1\}^{4m}$

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 |

# Reduction: $S(F) \circ g \le \mathrm{mKW}(\textsc{Xor-Sat})$

**Alice**

$x \in [m]^4$



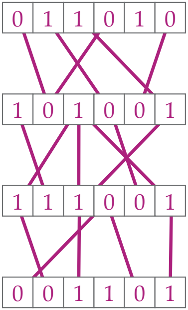

$\in \textsc{Xor-Sat}^{-1}(1)$

unsat $F$

add all constraints satisfied by $y$

**Bob**

$y \in \{0,1\}^{4m}$




$\in \textsc{Xor-Sat}^{-1}(0)$

# Query-to-communication lifting

$$
\begin{array}{ccc}
\mathbf{PLS}^{\mathsf{cc}}(S(F)\circ g) & \overset{\text{[GGKS'18]}}{=} & \mathbf{PLS}^{\mathsf{dt}}(S(F)) \\
\rotatebox{90}{$\leq$} & & \parallel \\
\mathbf{PLS}^{\mathsf{cc}}(\mathrm{mKW}(f)) & & \text{Resolution width of } F \\
\parallel & & \uparrow \\
\log \mathrm{monCkt}(f) & & \text{unsat } \mathcal{C}\text{-CSP } F \\
\downarrow & & \\
\mathcal{C}\text{-}\textsc{Sat} \text{ is hard} & &
\end{array}
$$

# Total **NP** Search Problems (**TFNP**)

# Communication **TFNP**

**TFNP**$^{\mathsf{cc}}$ $=$ Total two-party search problems that admit $\log^{O(1)}(n)$-cost non-deterministic protocol

*Example:* $\mathrm{mKW}(f) \in$ **TFNP**$^{\mathsf{cc}}$ for every monotone $f$

# Communication **TFNP**

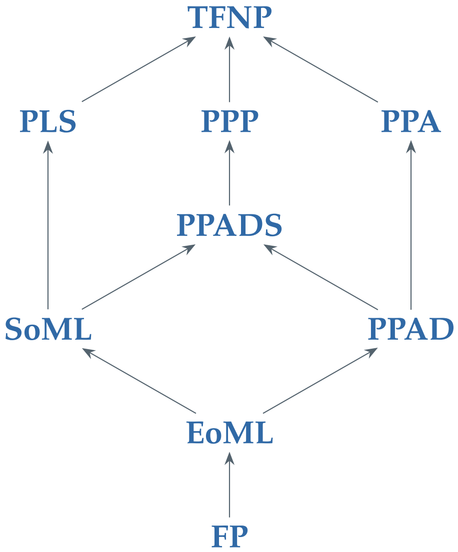
**TFNP**$^{\text{cc}}$ = Total two-party search problems that admit $\log^{O(1)}(n)$-cost non-deterministic protocol

*Example:* $\mathrm{mKW}(f) \in$ **TFNP**$^{\text{cc}}$ for every monotone $f$

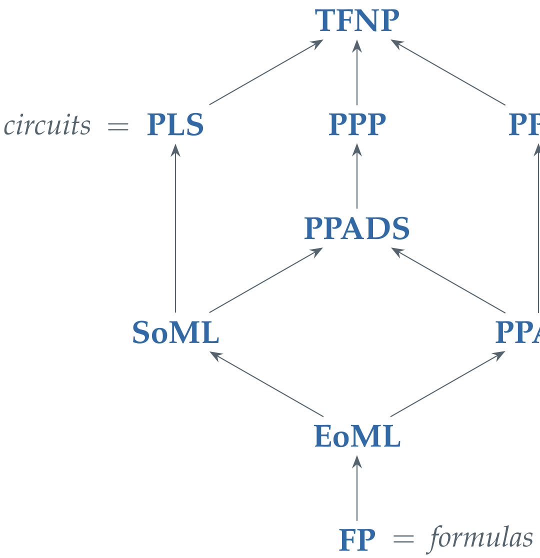
*Converse!* $\forall S \in$ **TFNP**$^{\text{cc}}$ there is $2^{\log^{O(1)}(n)}$-bit **partial** monotone $f$ with $S \leq \mathrm{mKW}(f)$

(*Proof:* reduce to set-disjointness + flip Bob's bits)

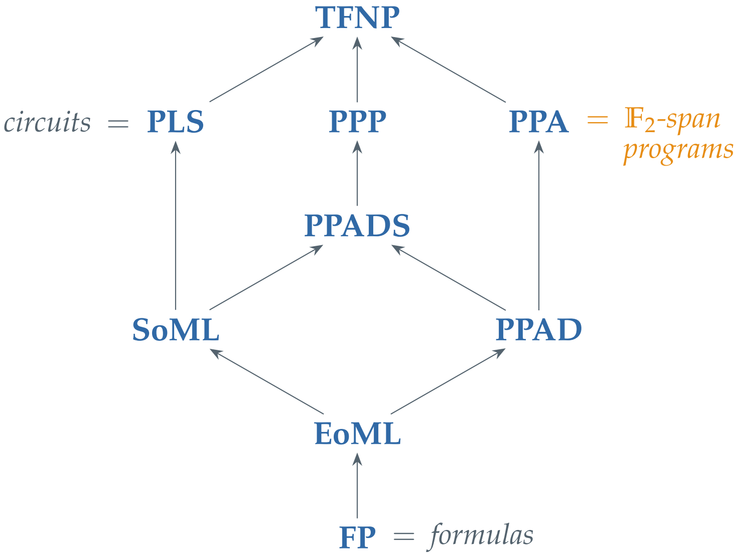
# Communication **TFNP** — What we know



# Communication **TFNP** — What we know



# Communication **TFNP** — What we know



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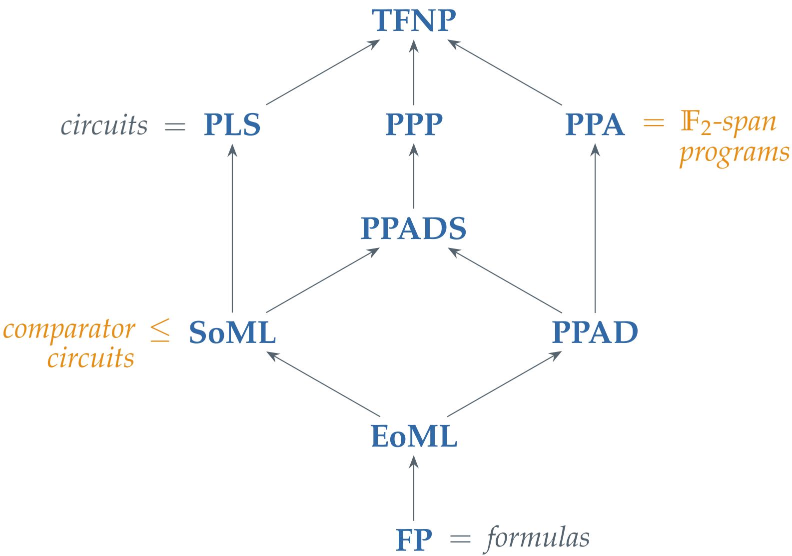

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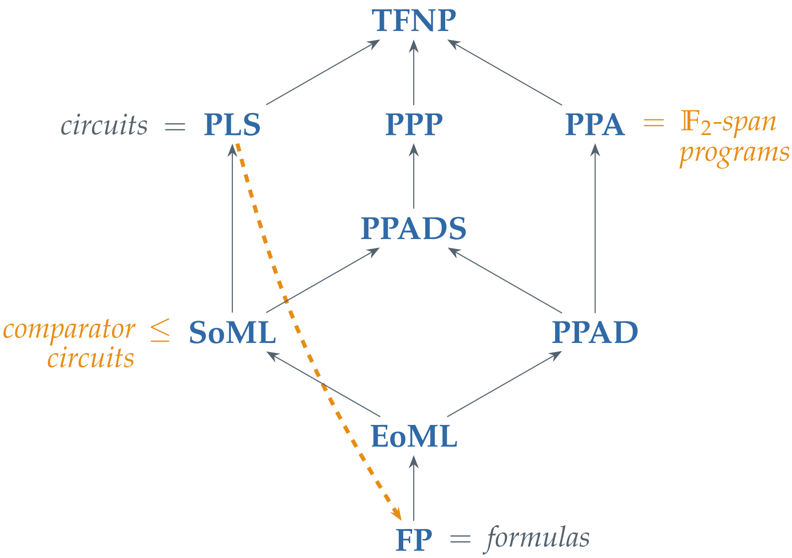

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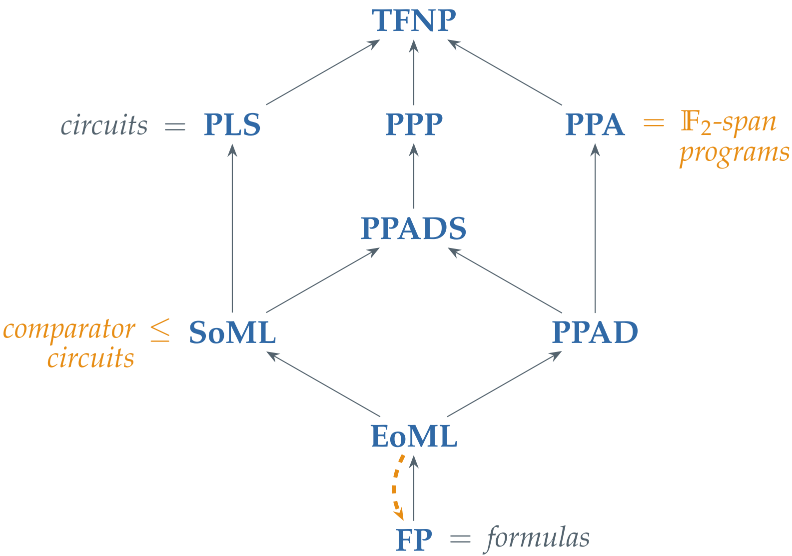

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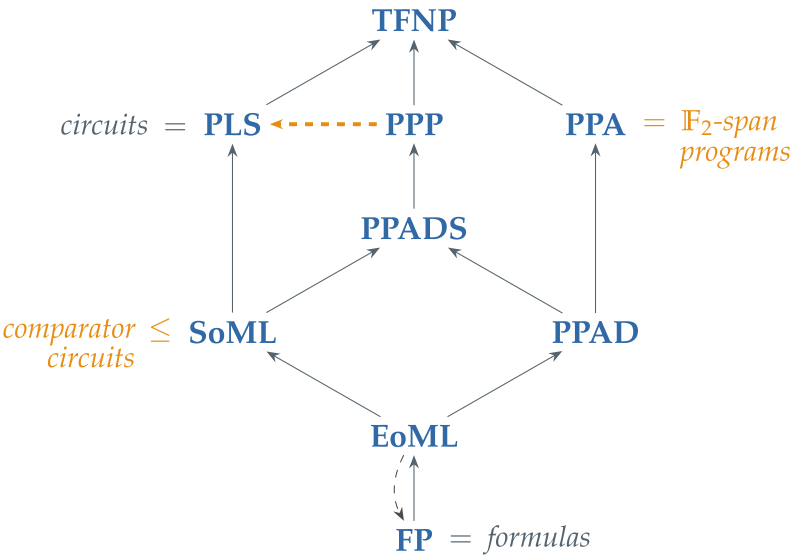

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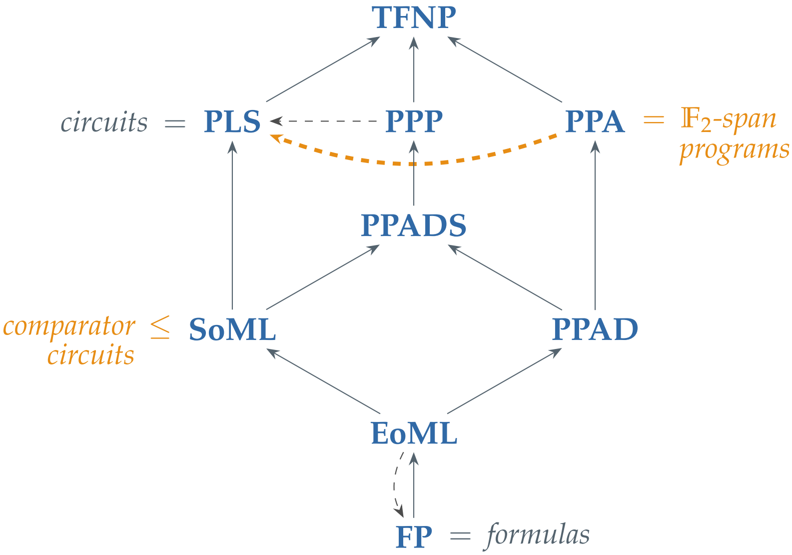

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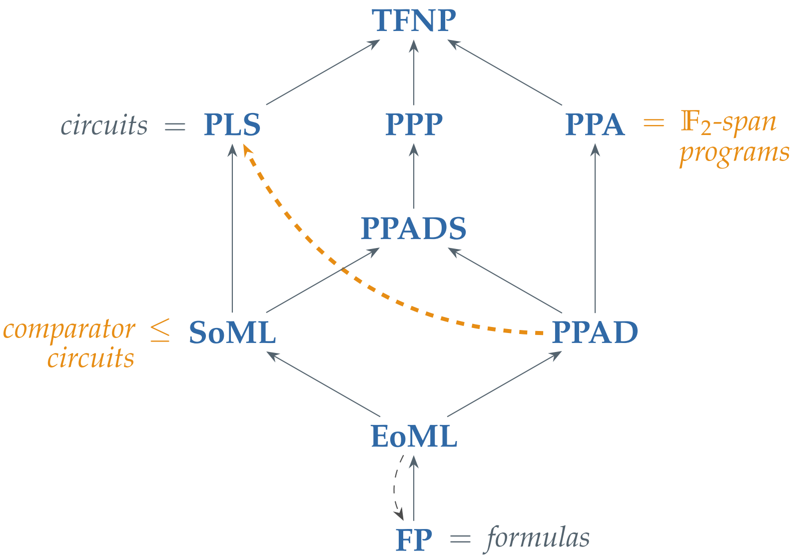

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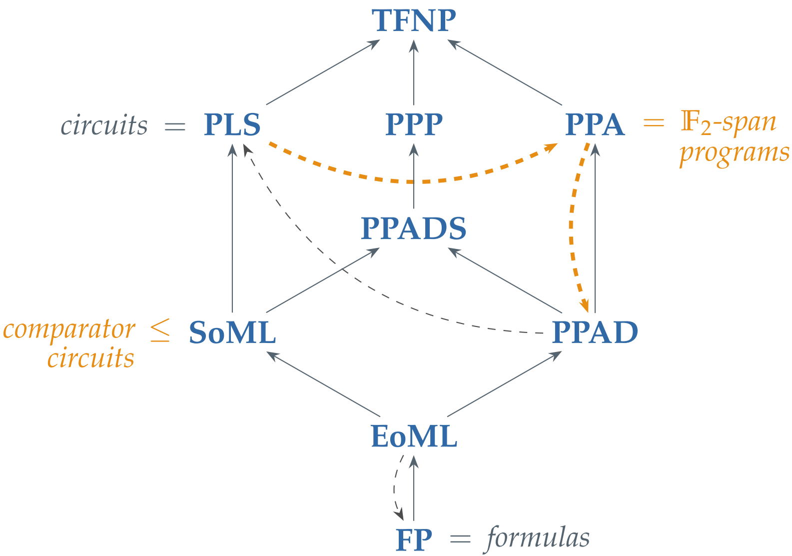

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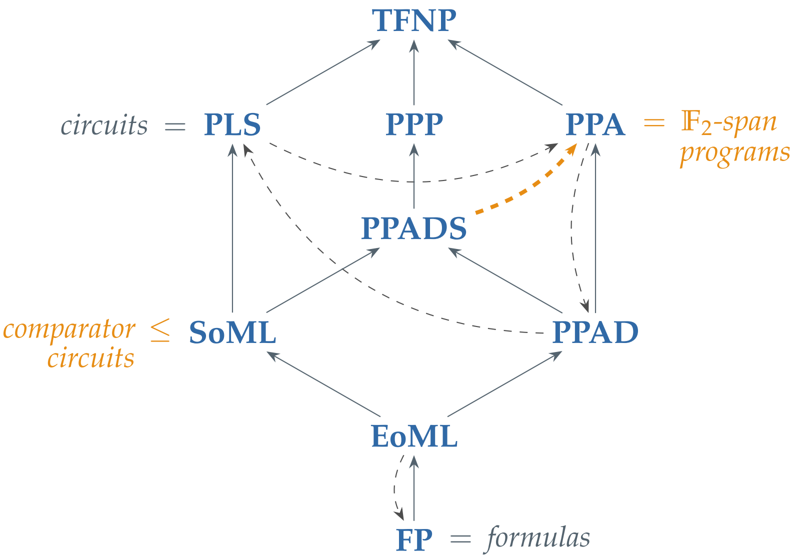

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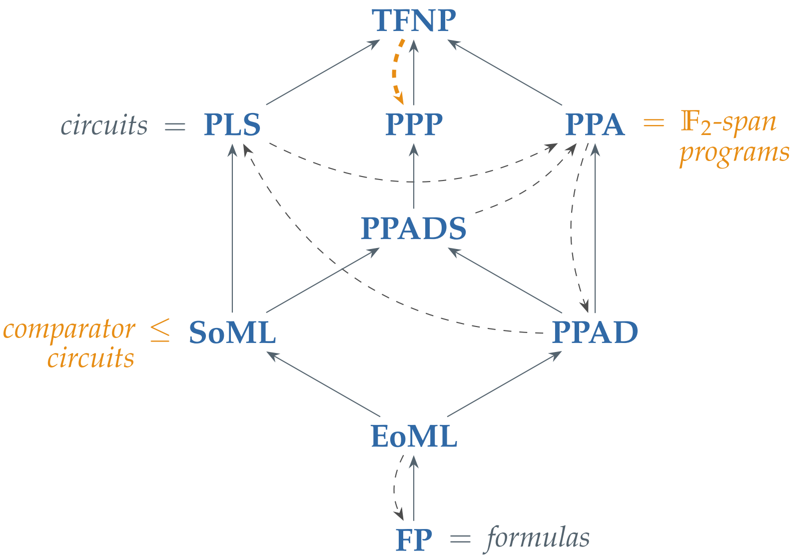
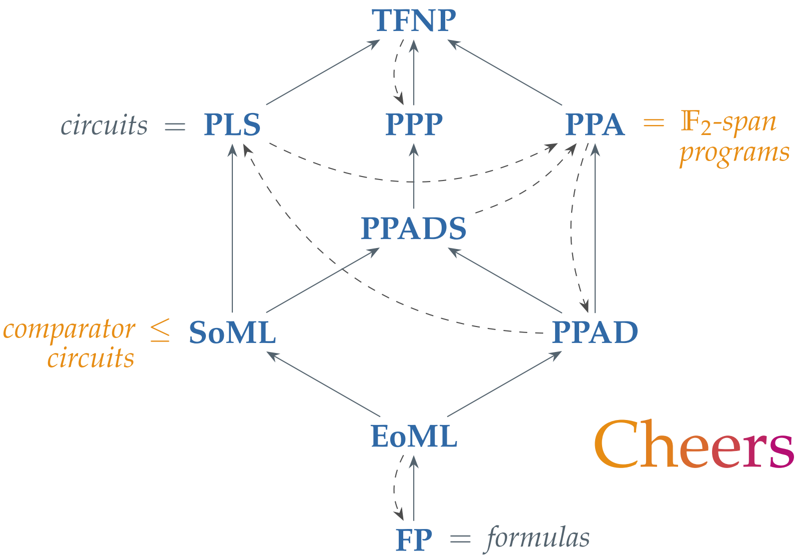


# Communication **TFNP** — What we know




Cheers!