Lecture 06

Foundations of Probabilistic Proofs
Fall 2020
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Inefficiency of Honest Provers

Our focus so far: achieve a polynomial-time verifier.

What about the honest prover?

Say we are given a boolean formula $\phi(x_1,\dots,x_n)$.

- in the sumcheck protocol (for #SAT): $\text{time}(P)=\Omega(2^n|\phi|)$.
- in Shamir's protocol (for TQBF): $\text{time}(P)=\Omega(2^n|\phi|)$.

[in fact Shamir's original protocol, without Shen's simplification, reduced the QBF to a "simple" QBF, squaring #vars so $\text{time}(P)=O(2^{n^2}|\phi|)$]

Are these times useful for computations of interest?

Let M be a machine running in time T and space S, and define

$$L_M := \{x \mid M(x)=1\}.$$

The reduction from L_M to TQBF yields a boolean formula $\phi(x_1,\dots,x_n)$ with

$$n \geq (\log T)\cdot S$$

Even if $T,S=\text{poly}(n)$, the honest prover runs in time $\Omega(2^n)=\Omega(T^S)=n^{\omega(1)}$.

Doubly-Efficient Interactive Proofs

New goal: additionally restrict honest prover to run in polynomial time.

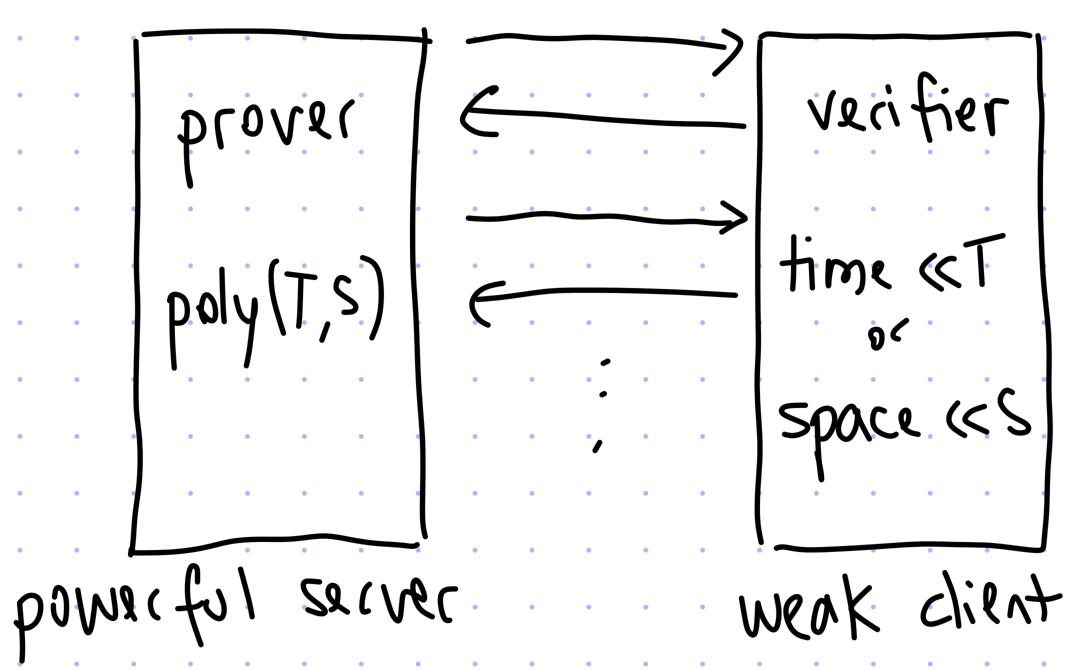
We call this a doubly-efficient interactive proof (deIP).

<u>claim:</u> $\mathrm{deIP} \subseteq \mathrm{BPP}$

<u>proof:</u> The probabilistic algorithm simulates the interaction between the honest prover and the honest verifier. $\square$

To make deIP non-trivial, we require the verifier to work less than deciding the language alone (e.g. less space, less time, ...).

This setting can be viewed as delegation of computation:



Q: what languages have doubly-efficient interactive proofs?

Delegation for Bounded-Depth Circuits

<u>theorem:</u> Suppose that L is decidable by $O(\log S)$-space uniform circuits of size S and depth D. Then L has a public-coin IP s.t.

- prover time is $\mathrm{poly}(S)$
- verifier time is $(n+D)\cdot\mathrm{polylog}\,S$ [& space is $O(\log S)$]
- communication (and # rounds) is $D\cdot\mathrm{polylog}(S)$.

<u>Note:</u> a circuit family $\{C_n\}_{n\in\mathbb{N}}$ is S-space uniform if there exists a machine M s.t. $M(1^n)=C_n$ runs in space $O(S(n))$

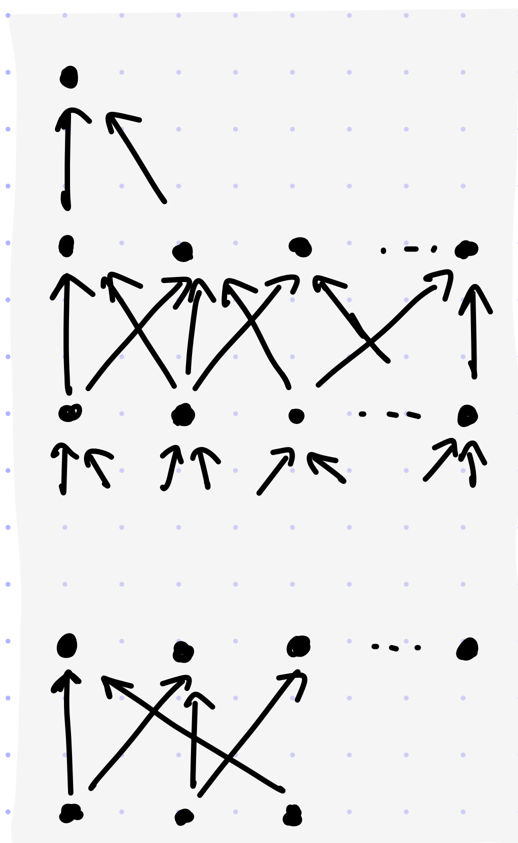
The proof of the theorem is quite technical.
We will see one piece: the "bare bones" protocol, which is same as above except that the verifier has oracle access to information about the circuit's topology (it will make $O(D)$ calls to this oracle) which saves us from discussing uniformity.

Main tools for bare-bones protocol: this has been implemented and it is very efficient!

more arithmetization, more sumcheck, some new ideas.

Layered Arithmetic Circuits

A layered arithmetic circuit $C: \mathbb{F}^n \to \mathbb{F}$ of size S and depth D (with $n \leq S$) is an arithmetic circuit with fan-in 2 arranged in $D+1$ layers:



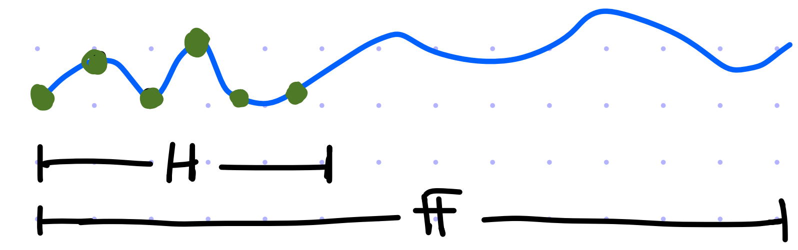
layer 0	1 output	$V_0 \in \mathbb{F}$	
layer 1	S internal values	$V_1: [S] \to \mathbb{F}$	$add_1: [S]^3 \to \{0,1\}$, $mul_1: [S]^3 \to \{0,1\}$
layer 2	S internal values	$V_2: [S] \to \mathbb{F}$	$add_2: [S]^3 \to \{0,1\}$, $mul_2: [S]^3 \to \{0,1\}$
⋮		⋮	
layer D-1	S internal values	$V_{D-1}: [S] \to \mathbb{F}$	
layer D	n inputs	$V_D: [n] \to \mathbb{F}$	$add_D: [S] \times [n]^2 \to \{0,1\}$, $mul_D: [S] \times [n]^2 \to \{0,1\}$

The wiring predicates $[(add_i, mul_i)]_{i=1,\dots,D}$ describe the circuit C: add_i/mul_i at (a,b,c) is 1 if a-th value in layer $i-1$ is the addition/multiplication of b-th & c-th values in layer i.

For notational simplicity, we assume C has 1 type of gate: $g: \mathbb{F}^2 \to \mathbb{F}$.

We can take $(wp_1, \dots, wp_D)$ to be the wiring predicates for g. [Extending to multiple gate types is straightforward.]

Low-Degree Extension



Let $H \subseteq \mathbb{F}$ be a domain, and $f : H \to \mathbb{F}$ a function.

A polynomial $p \in \mathbb{F}[X]$ is an extension of f if $p|_H \equiv f$.

It is a low-degree extension if p has "low degree" [the specific condition varies].

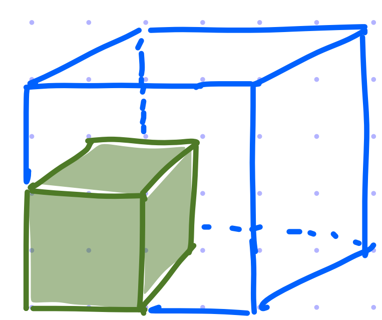
The higher the allowed degree, the more low-degree extensions a function has.

The minimal degree one is unique: it has degree $< |H|$ and equals

$$p(X) := \sum_{\alpha \in H} f(\alpha) \cdot L_{H,\alpha}(X) = \sum_{\alpha \in H} f(\alpha) \cdot \left(\prod_{\beta \in H \setminus \{\alpha\}} \frac{X - \beta}{\alpha - \beta} \right).$$

The polynomials $\{L_{H,\alpha}(X)\}_{\alpha \in H}$ are the Lagrange polynomials.

The multivariate case is straightforward:



- $p \in \mathbb{F}[X_1, \dots, X_n]$ extends $f : H^n \to \mathbb{F}$ if $p|_{H^n} \equiv f$.
- f's extension of minimal degree has individual degree $< |H|$ and equals

$$p(X_1, \dots, X_n) := \sum_{\alpha_1, \dots, \alpha_n \in H} f(\alpha_1, \dots, \alpha_n) \cdot L_{H^n, \alpha_1, \dots, \alpha_n}(X_1, \dots, X_n) = \sum_{\alpha_1, \dots, \alpha_n \in H} f(\alpha_1, \dots, \alpha_n) \cdot \left(\prod_{i \in [n]} L_{H, \alpha_i}(X_i) \right).$$

Arithmetize Each Layer

Fix a subset $H \subseteq \mathbb{F}$ of size $\log S$ and set $m := \frac{\log S}{\log|H|}$ and $m_{in} := \frac{\log n}{\log|H|}$.

This induces bijections $[S] \leftrightarrow H^m$ and $[n] \leftrightarrow H^{m_{in}}$.

Step 1: rewrite computation as summations

Let $z \in \mathbb{F}^n$ be an input to the circuit $C: \mathbb{F}^n \to \mathbb{F}$.

- the input layer $V_D: H^{m_{in}} \to \mathbb{F}$ is defined as $V_D(a) := z_a$
- for $i = D-1, \dots, 1, 0$: $V_i: H^m \to \mathbb{F}$ is defined as $V_i(a) := \sum_{b,c \in H^m} wp_{i+1}(a,b,c) \cdot g(V_{i+1}(b), V_{i+1}(c))$

Step 2: low-degree extend each layer

- the extension of the input layer is $\hat{V}_D: \mathbb{F}^{m_{in}} \to \mathbb{F}$ where

$$\hat{V}_D(X) := \sum_{a \in H^{m_{in}}} z_a \cdot L_{H^{m_{in}},a}(X).$$

- the extension of the i-th layer is $\hat{V}_i: \mathbb{F}^m \to \mathbb{F}$ where

$$\hat{V}_i(x) := \sum_{a \in H^m} \left(\sum_{b,c \in H^m} \hat{wp}_{i+1}(a,b,c) \cdot g(\hat{V}_{i+1}(b), \hat{V}_{i+1}(c)) \right) \cdot L_{H^m,a}(X).$$

can consider extension instead of function as summation is over H

analogous to Shen's relinearization

Step 3: replace $L_{H^m,a}(X)$ with $I_{H^m}(X,a)$ where $I_{H^m}(X,Y) := \prod_{i=1}^{m} \sum_{\alpha \in H} L_{H,\alpha}(X_i) L_{H,\alpha}(Y_i)$
to ensure that a has low degree in addend

Rewrite Computation as Iterated Sumchecks

The statement "$C(z) = y$" is rewritten as "$\hat{V}_0(0) = y$".

Equivalently, "$\sum_{a,b,c \in H^m} \hat{wp}_1(a,b,c) \cdot g(\hat{V}_1(b), \hat{V}_1(c)) \cdot I_{H^m}(0,a) = y$".

So we can do a sumcheck on variables for a,b,c. This involves:

- $3m$ rounds [we are summing over the hypercube H^{3m}]
- soundness error $O\left(m \cdot \frac{|H|}{|\mathbb{F}|}\right)$ [individual degrees are $O(|H|)$ so we pay $O\left(\frac{|H|}{|\mathbb{F}|}\right)$ per round]
- $\mathrm{poly}(|H|^m) = \mathrm{poly}(S)$ operation for the honest prover — THIS IS EFFICIENT
- $\mathrm{poly}(m, |H|) = \mathrm{poly}(\log S)$ operations for the verifier, given
  - 1 query to $\hat{wp}_1 : \mathbb{F}^{3m} \to \mathbb{F}$ ← assume that verifier can evaluate on its own
  - 2 queries to $\hat{V}_1 : \mathbb{F}^m \to \mathbb{F}$

The prover sends the answers and we recurse on two claims: "$\hat{V}_1(s) = \gamma$" and "$\hat{V}_1(t) = \delta$".

Indeed, each claim is itself a sum:

"$\sum_{a,b,c \in H^m} \hat{wp}_2(a,b,c) \cdot g(\hat{V}_2(b), \hat{V}_2(c)) \cdot I_{H^m}(s,a) = \gamma$"

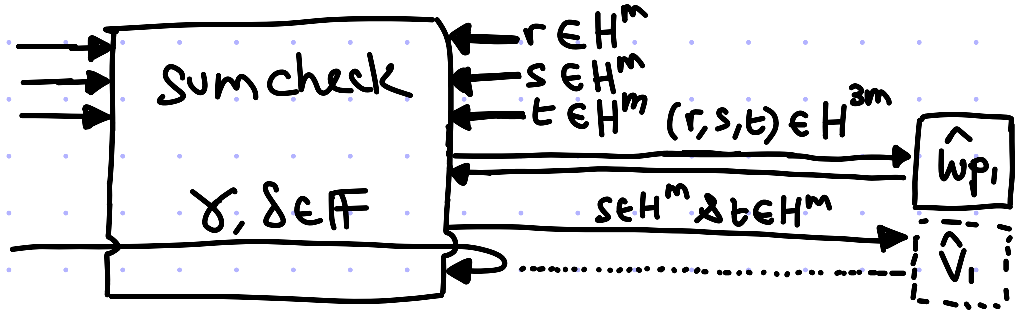
"$\sum_{a,b,c \in H^m} \hat{wp}_2(a,b,c) \cdot g(\hat{V}_2(b), \hat{V}_2(c)) \cdot I_{H^m}(t,a) = \delta$"

Problem: the number of claims doubles at each layer

Avoiding Claim Blowup

1 claim about layer 0

$$\text{“}\sum_{a,b,c\in H^m} \widehat{wp}_1(a,b,c)\cdot g(\hat{V}_1(b),\hat{V}_1(c))\cdot I_{H^m}(0,a) = y\text{”}.$$



2 claims about layer 1

$$\text{“}\sum_{a,b,c\in H^m} \widehat{wp}_2(a,b,c)\cdot g(\hat{V}_2(b),\hat{V}_2(c))\cdot I_{H^m}(s,a) = \gamma\text{”}$$

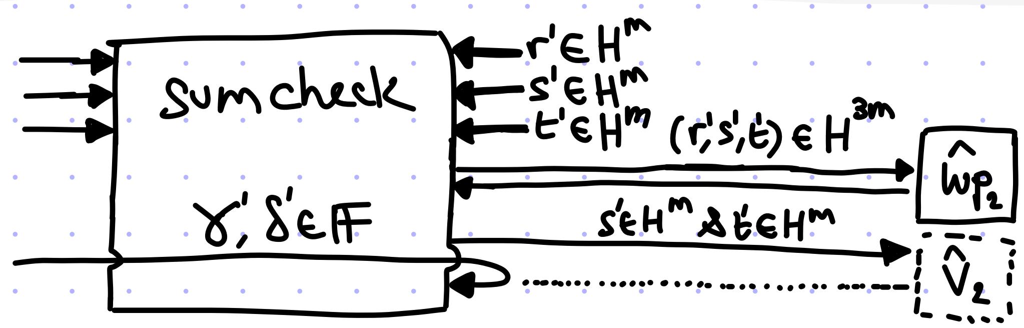
$$\text{“}\sum_{a,b,c\in H^m} \widehat{wp}_2(a,b,c)\cdot g(\hat{V}_2(b),\hat{V}_2(c))\cdot I_{H^m}(t,a) = \delta\text{”}$$

$\xleftarrow{\alpha,\beta\in_R\mathbb{F}}$

1 claim about layer 1

$$\text{“}\sum_{a,b,c\in H^m} \widehat{wp}_2(a,b,c)\cdot g(\hat{V}_2(b),\hat{V}_2(c))\cdot\left[\alpha\cdot I_{H^m}(s,a)+\beta\cdot I_{H^m}(t,a)\right] = \alpha\cdot\gamma+\beta\cdot\delta\text{”}$$

random linear combination of the 2 claims



2 claims about layer 2

$$\text{“}\sum_{a,b,c\in H^m} \widehat{wp}_3(a,b,c)\cdot g(\hat{V}_3(b),\hat{V}_3(c))\cdot I_{H^m}(s',a) = \gamma'\text{”}$$

$$\text{“}\sum_{a,b,c\in H^m} \widehat{wp}_3(a,b,c)\cdot g(\hat{V}_3(b),\hat{V}_3(c))\cdot I_{H^m}(t',a) = \delta'\text{”}$$

… and so on.

Protocol Summary

- public coin
- number of rounds is

  $D \cdot (\text{sumcheck on } 3m \text{ vars} + 1)$

  $= O(Dm) = O(D \cdot \log S)$
- communication complexity (in elts) is:

  $D \cdot (\text{sumcheck on } 3m \text{ vars of deg } O(|H|) + 2)$

  $= O(D \cdot m \cdot |H|) = O(D \,\text{polylog} S).$
- soundness error is

  $D \cdot \left(\text{sumcheck on } 3m \text{ vars of deg } O(|H|) + \frac{1}{|\mathbb{F}|}\right)$

  $= O\left(\frac{D \cdot m \cdot |H|}{|\mathbb{F}|}\right) \Rightarrow |\mathbb{F}| \geq D \,\text{polylog} S$ suffices
- prover time (in field operations) is

  $D \cdot (\text{sumcheck on } 3m \text{ vars of deg } O(|H|))$

  $= D \cdot \text{poly}(|H|^m) = \text{poly}(S)$
- verifier time (in field operations) is

  $O(D \cdot \text{poly}(m, |H|)) = D \,\text{polylog} S$ [+ eval each of $\hat{wp}_1, \ldots, \hat{wp}_D$ at one location]

1 claim about $\hat{V}_0$

1 query to $\hat{wp}_1$ — sumcheck

2 claims about $\hat{V}_1$

combination

1 claim about $\hat{V}_1$

1 query to $\hat{wp}_2$ — sumcheck

2 claims about $\hat{V}_2$

combination

1 claim about $\hat{V}_2$

⋮

1 claim about $\hat{V}_D$

[can compute on its own]

Did not discuss:

in last sumcheck one variable group is in $H^{m_{in}}$ [not H^m]