Lecture 02

Foundations of Probabilistic Proofs
Fall 2020
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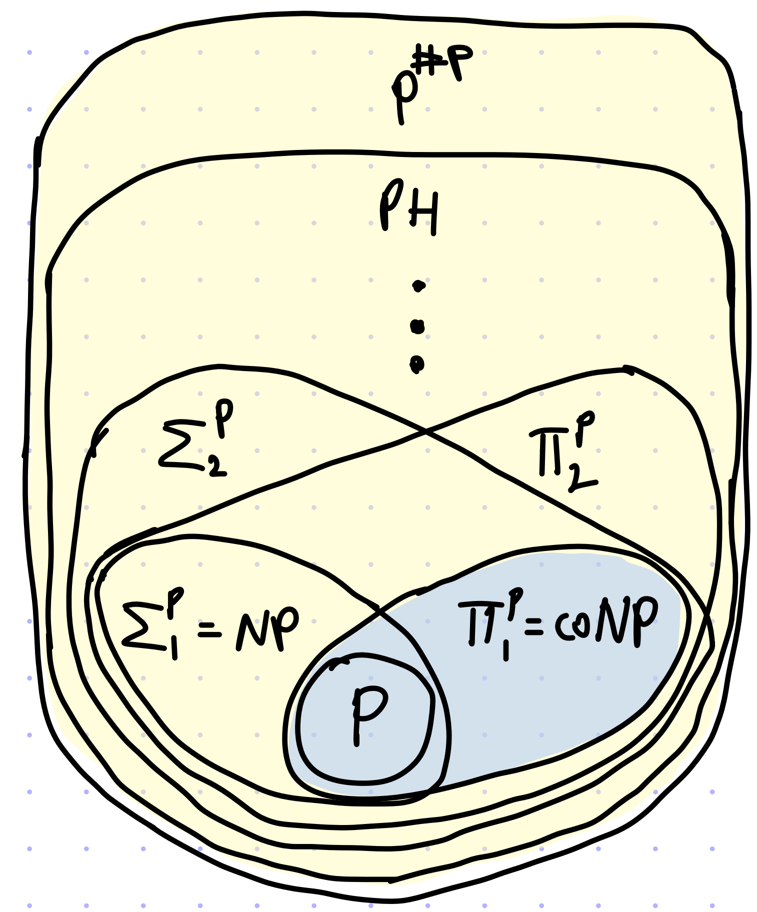
Interactive Proofs for Counting Problems

We saw an interactive proof for GNI, a problem in coNP not known to be in P.
Yet, GNI is not believed to be coNP-complete. [If so, PH collapses to 2nd level.]

theorem: UNSAT $\in$ IP, so $coNP \subseteq IP$

theorem: #SAT $\in$ IP, so $P^{\#P} \subseteq IP$



These results should be surprising:

- many languages beyond NP!
- the interactive proof for GNI leveraged properties of graph isomorphisms, but UNSAT and #SAT do not seem to have similar properties

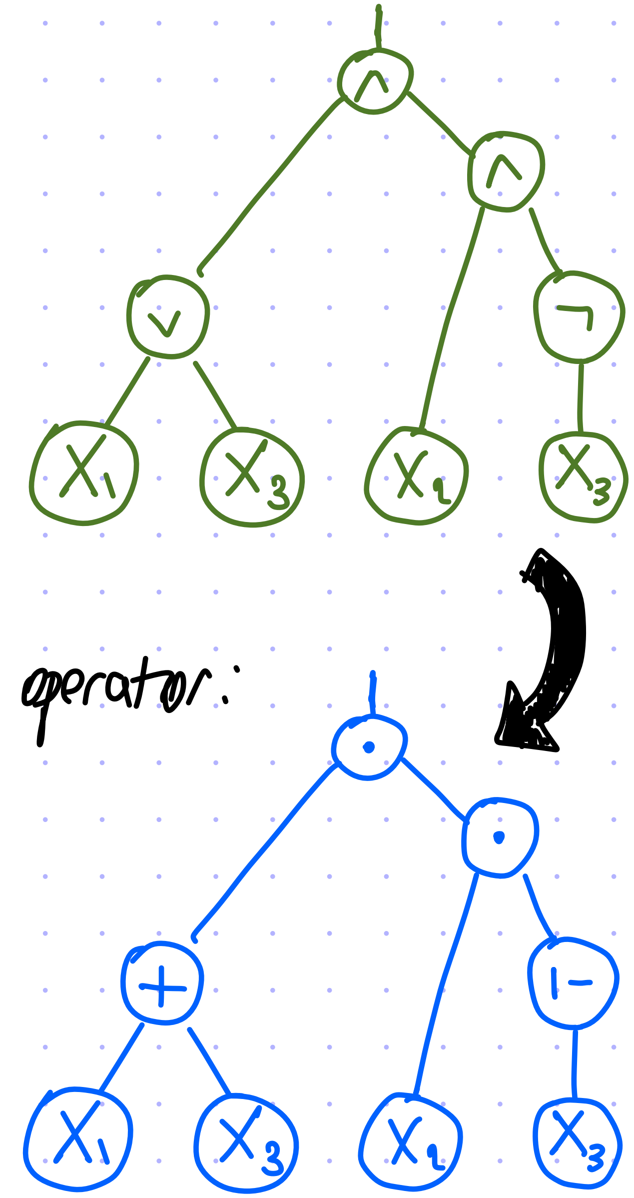
$\Rightarrow$ we will learn new ideas: arithmetization, sumcheck protocol

[$P^{\#P}$ = languages decidable in polynomial time via a machine with a #SAT oracle]

Arithmetization of a Boolean Formula

A boolean formula $\phi(x_1,\dots,x_n)$ is a tree where:

- every leaf node is labeled by a variable x_i;
- every internal node is a logical operator on its children. $(\vee, \wedge, \neg)$



Arithmetization replaces each logical operator with an arithmetic operator:

$$\neg X \mapsto 1-X \qquad X \wedge Y \mapsto X \cdot Y \qquad X \vee Y \mapsto X+Y$$

Thus a boolean formula $\phi(x_1,\dots,x_n)$ is mapped to a polynomial $p(x_1,\dots,x_n)$ such that $\deg_{tot}(p) \le |\phi|$, evaluating p at a point takes $|\phi|$ operations, and:

<u>claim:</u> Let ϕ be a 3CNF. Then:

- $\phi \in \text{UNSAT} \Rightarrow \displaystyle\sum_{a_1,\dots,a_n \in \{0,1\}} p(a_1,\dots,a_n) = 0$
- $\phi \notin \text{UNSAT} \Rightarrow 0 < \displaystyle\sum_{a_1,\dots,a_n \in \{0,1\}} p(a_1,\dots,a_n) \le 2^n \cdot 3^m$

$\Rightarrow$

<u>corollary:</u>

$\forall$ prime $q > 2^n 3^m$

$$\phi \in \text{UNSAT} \iff \sum_{a_1,\dots,a_n \in \{0,1\}} p(a_1,\dots,a_n) = 0 \mod q$$

Sumcheck Protocol

statement

$$\sum_{\alpha_1,\dots,\alpha_n \in H} p(\alpha_1,\dots,\alpha_n) = \gamma$$

$P(\mathbb{F}, H, n, \gamma, p)$ $\qquad$ $V^{p}(\mathbb{F}, H, n, \gamma)$

$P(\mathbb{F}, H, n, \gamma, p)$		$V^{p}(\mathbb{F}, H, n, \gamma)$
$p_1(X) := \sum_{\alpha_2,\dots,\alpha_n \in H} p(X, \alpha_2, \dots, \alpha_n)$	$\xrightarrow{p_1 \in \mathbb{F}[X]}$	$\sum_{\alpha_1 \in H} p_1(\alpha_1) \stackrel{?}{=} \gamma$
	$\xleftarrow{\omega_1 \in \mathbb{F}}$	$w_1 \leftarrow \mathbb{F}$
$p_2(X) := \sum_{\alpha_3,\dots,\alpha_n \in H} p(w_1, X, \alpha_3, \dots, \alpha_n)$	$\xrightarrow{p_2 \in \mathbb{F}[X]}$	$\sum_{\alpha_2 \in H} p_2(\alpha_2) \stackrel{?}{=} p_1(w_1)$
	$\xleftarrow{w_2 \in \mathbb{F}}$	$w_2 \leftarrow \mathbb{F}$
	$\vdots$	$\vdots$
$p_n(X) := p(w_1, \dots, w_{n-1}, X)$	$\xrightarrow{p_n \in \mathbb{F}[X]}$	$\sum_{\alpha_n \in H} p_n(\alpha_n) = p_{n-1}(w_{n-1})$
	$\xleftarrow{\omega_n \in \mathbb{F}}$	$w_n \leftarrow \mathbb{F}$
		$p(w_1, \dots, w_n) \stackrel{?}{=} p_n(w_n)$
	$O(n \cdot \deg_{ind}(p))$ elements	$O(n \cdot \lvert H \rvert \cdot \deg_{ind}(p))$ fops $+ 1$ eval of p

claim: if statement is true then verifier accepts w.p. 1

Soundness of Sumcheck Protocol

can improve to $1-\left(1-\frac{\deg_{ind}(p)}{|\mathbb{F}|}\right)^n$

<u>claim</u>: if $\sum_{\alpha_1,\ldots,\alpha_n \in H} p(\alpha_1,\ldots,\alpha_n) \neq \gamma$ then $\Pr[\text{verifier accepts}] \leq \frac{n \cdot \deg_{ind}(p)}{|\mathbb{F}|}$

<u>proof</u>: Fix a malicious prover, described via n polynomials $\tilde{p}_1,\ldots,\tilde{p}_n \in \mathbb{F}[X]$ such that $\tilde{p}_i$ depends on the verifier's messages $\omega_1,\ldots,\omega_{i-1} \in \mathbb{F}$.

Define: $\forall i \in [n]$ $E_i :=$ "event that $\tilde{p}_i \equiv p_i$", $W =$ "event that verifier accepts".

<u>lemma</u>: For $j = n, n-1, \ldots, 1$: $\Pr[W] \leq \frac{(n-j+1)\cdot \deg_{ind}(p)}{|\mathbb{F}|} + \Pr[W \mid E_j \wedge \cdots \wedge E_n]$.

This suffices to prove the claim because if we set $j:=1$ then we get:

$$\Pr[W] \leq \frac{n \cdot \deg_{ind}(p)}{|\mathbb{F}|} + \underbrace{\Pr[W \mid E_1 \wedge \cdots \wedge E_n]}$$

$\leq \Pr[W \mid E_1] = 0$ because $\sum_{\alpha_1 \in H} \tilde{p}_1(\alpha_1) = \sum_{\alpha_1 \in H} p_1(\alpha_1) \neq \gamma$

$$= \frac{n \cdot \deg_{ind}(p)}{|\mathbb{F}|} + 0$$

We are left to prove the lemma.

Lemma: For $j = n, n-1, \ldots, 1$: $\Pr[W] \leq \dfrac{(n-j+1)\cdot \deg_{ind}(p)}{|\mathbb{F}|} + \Pr[W \mid E_j \wedge \cdots \wedge E_n]$.

Proof is by induction on j.

Base case: $j = n$

$$\Pr[W] \leq \Pr[W \mid \overline{E_n}] + \Pr[W \mid E_n] \leq \frac{\deg_{ind}(p)}{|\mathbb{F}|} + \Pr[W \mid E_n]$$

$$= \Pr[V \text{ accepts} \mid \tilde{p}_n \not\equiv p_n]$$

$$\leq \Pr[\tilde{p}_n(w_n) = p(w_1, \ldots, w_n) \mid \tilde{p}_n \not\equiv p_n]$$

$$= \Pr[\tilde{p}_n(w_n) = p_n(w_n) \mid \tilde{p}_n \not\equiv p_n] \leq$$

Polynomial Identity Lemma

$$\Pr_{\alpha \leftarrow S}\left[f \in \mathbb{F}[X] \text{ vanishes at } \alpha\right] \leq \frac{\deg(f)}{|S|}$$

Inductive case:

$$\leq \Pr\Big[\tilde{p}_{j-1}(w_{j-1}) = \sum_{\alpha_j \in H} \tilde{p}_j(\alpha_j) \;\Big|\; \tilde{p}_{j-1} \not\equiv p_{j-1}, \tilde{p}_j \equiv p_j\Big]$$

$$\leq \Pr\Big[\tilde{p}_{j-1}(w_{j-1}) = \sum_{\alpha_j \in H} p_j(\alpha_j) \;\Big|\; \tilde{p}_{j-1} \not\equiv p_{j-1}\Big]$$

$$= \Pr[\tilde{p}_{j-1}(w_{j-1}) = p_{j-1}(w_{j-1}) \mid \tilde{p}_{j-1} \not\equiv p_{j-1}]$$

assume for $j \in \{n, n-1, \ldots, 2\}$

$$\Pr[W] \leq \frac{(n-j+1)\cdot \deg_{ind}(p)}{|\mathbb{F}|} + \Pr[W \mid E_j \wedge \cdots \wedge E_n]$$

$$\leq \frac{(n-j+1)\cdot \deg_{ind}(p)}{|\mathbb{F}|} + \Pr[W \mid \overline{E_{j-1}} \wedge E_j \wedge \cdots \wedge E_n] + \Pr[W \mid E_{j-1} \wedge E_j \wedge \cdots \wedge E_n]$$

$$\leq \frac{(n-j+1)\cdot \deg_{ind}(p)}{|\mathbb{F}|} + \frac{\deg_{ind}(p)}{|\mathbb{F}|} + \Pr[W \mid E_{j-1} \wedge \cdots \wedge E_n]$$

proved for $j-1$

$$\leq \frac{(n-(j-1)+1)\cdot \deg_{ind}(p)}{|\mathbb{F}|} + \Pr[W \mid E_{j-1} \wedge \cdots \wedge E_n]$$

Interactive Proof for UNSAT (which shows that $coNP \subseteq IP$)

$P(\phi)$ | the 3CNF ϕ is unsatisfiable | $V(\phi)$

$$P(\phi) \xrightarrow{q \in \mathbb{N}} V(\phi)$$

$2^n 3^m < q < 2^{poly(m,n)}$

$q \in$ PRIMES [probabilistic test suffices]

$p := \text{ARITH}(\phi, \mathbb{F}_q)$ $\qquad\qquad$ $p := \text{ARITH}(\phi, \mathbb{F}_q)$

$P_{sc}(\mathbb{F}_q, \{0,1\}, n, 0, p)$

sumcheck protocol

$$\sum_{\alpha_1,\dots,\alpha_n \in \{0,1\}} p(\alpha_1,\dots,\alpha_n) \stackrel{?}{=} 0$$

$V_{sc}^{p}(\mathbb{F}_q, \{0,1\}, n, 0)$

$(\omega_1,\dots,\omega_n) \in \mathbb{F}_q^n$ $\qquad$ $p(\omega_1,\dots,\omega_n) \in \mathbb{F}_q$

evaluate p at $(\omega_1,\dots,\omega_n)$ in $poly(m,n)$ time

Arithmetization for #SAT

The arithmetization we used for UNSAT was coarse:

$$\forall (a_1,\dots,a_n) \in \{0,1\}^n \quad \begin{array}{l} \phi(a_1,\dots,a_n) = \text{false} \rightarrow p(a_1,\dots,a_n) = 0 \\ \phi(a_1,\dots,a_n) = \text{true} \rightarrow 0 < p(a_1,\dots,a_n) \leq 3^m \end{array}$$

We can modify the arithmetization to be more precise:

$$\neg x \mapsto 1-x \qquad x \wedge y \mapsto x \cdot y \qquad x \vee y \mapsto x+y-x\cdot y$$

The new arithmetization satisfies:

claim: $\forall (a_1,\dots,a_n) \in \{0,1\}^n \quad \begin{array}{l} \phi(a_1,\dots,a_n) = \text{false} \rightarrow p(a_1,\dots,a_n) = 0 \\ \phi(a_1,\dots,a_n) = \text{true} \rightarrow p(a_1,\dots,a_n) = 1 \end{array}$

We can now reduce #SAT to a sumcheck problem:

corollary: $\forall$ prime $q > 2^n$ $\quad \#\phi = c \iff \sum_{a_1,\dots,a_n \in \{0,1\}} p(a_1,\dots,a_n) = c \mod q$

Interactive Proof for #SAT

(which shows that $P^{\#P} \subseteq IP$)

$P(\phi)$

$\#\phi = c$

$V(\phi)$

$\xrightarrow{q \in \mathbb{N}}$

$2^n < q < 2^{poly(m,n)}$

$q \in PRIMES$

$p := ARITH^{\#}(\phi, \mathbb{F}_q)$

$p := ARITH^{\#}(\phi, \mathbb{F}_q)$

$P_{sc}(\mathbb{F}_q, \{0,1\}, n, c, p)$

sumcheck protocol

$$\sum_{\alpha_1, \dots, \alpha_n \in \{0,1\}} p(\alpha_1, \dots, \alpha_n) \stackrel{?}{=} c$$

$V_{sc}^{p}(\mathbb{F}_q, \{0,1\}, n, c)$

$(w_1, \dots, w_n) \in \mathbb{F}_q^n$

$p(w_1, \dots, w_n) \in \mathbb{F}_q$

evaluate p at $(w_1, \dots, w_n)$ in poly(m,n) time

Let $L \in P^{\#P}$, and let M be a machine that decides L with a #SAT oracle.

Here is the IP for L

$V(x)$

simulate M on x and ask prover for help on #SAT calls

ϕ

c

IP for $\#\phi = c$

$P(x)$